THEORY OF OPERATOR FIELDS I

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Honolulu, Hawaii

October 1, 1953

Supported by Office of Naval Research

(Contract No. 976(00), Project NR 010 106)

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ABSTRACT

Some results of the study of operators and their representation in terms of basic operators are presented. It is shown that the generalized expansion in terms of the basic functions satisfies a least square requirement which is independent of the representation of the operators. The notion of *representatives* and *matrix elements* of the *second kind* is introduced together with the idea of *transformation functions* of the *second kind.* A representation of operators is set forth very similar to the Fourier Integral expansion but with the Fourier coefficients manifested as functions of the displacement operator (3.27), (3.36) or as functions of the space-time operators (3.33), (3.35). A reduction of the solution of a linear operator equation to the solution of a partial differential equation is effected. Consideration is given to a definition of an averaging process for operator fields and the probability distribution of operators relative to the averaging process. The averaging process is also independent of the representation of the operators.

Operator identities are obtained by considering the invariance properties under tensor and/or similarity transformations of arbitrary functions of operators. These identities are used to study the question of the existence and structure of conservation equations for operator or non-local fields. In the limit of c- number fields conservation

equations exist but for operator fields another operator equation besides the field equations must be satisfied. The former is always satisfied for C- number fields. An example which leads to conservation equations is considered.

In order to make some connection of the theory with possible experiments some consequences of the interaction of a non-local photon with a constant electromagnetic E, H field are considered. The interaction is shown to vanish in the limit of local-fields independently of the magnitude of the coupling constant g so that the results obtained could be traced directly to the assumption of a non-local field. An exact solution for the non-local field is obtained as a sum of non-local plane waves containing the four vector q_μ. It is shown that this vector in conjunction with the coupling constant plays two roles. In one it is involved in the expression for an electric dipole moment: $g\underset{\sim}{q}$ and in the other in an expression for a magnetic dipole with moment $g/2(\underset{\sim}{q} \times k)/|k|$ where $\underset{\sim}{k}/|k|$ is the propagation vector. This identification gives significance to the C- numbers q_s appearing in the theory. The equivalent mass μ averaged over orientations of $\underset{\sim}{q}$ for which $\mu^2 > 0$ turns out to be of order $4.0 \times 10^{-12}(\gamma B/\lambda)^{1/2}$ electron mass, where γ is the number of Bohr magnetons: $|g\underset{\sim}{q}|\, mc/he$, λ the wave length and B an upper limit to the magnitude of the external field in Gaussian units. An order of magnitude masure of the optical properties of the region

containing the constant E, H field which affects the non-local photon is $|n-1| = 3.2 \times 10^{-4} (\lambda \gamma B)$, where n is an upper limit to the index of refraction. For pronounced measurable effects, say, $|n-1| \simeq .1$, $\langle\mu\rangle$ attains a value of $10^{-10}/\lambda$ electron masses.

I. INTRODUCTION

1. Notation and Fundamental Commutation Relationship

In this report we shall use the formalism described by Dirac in his treatise on Quantum Mechanics.[1] Throughout our investigation we will choose our units such as to render $\hbar \equiv h/2\pi = 1$, and $c = 1$, where h is Planck's constant and c the velocity of light. In view of this choice of units the commutation relationships between the contravariant space-time four-vector operators x^μ and the covariant four-vector energy-momentum vector p_μ are

$$[x^\mu, x^\nu] = [p_\mu, p_\nu] = 0, \qquad (1.1)$$
$$[p_\nu, x^\mu] = -i\,\delta^\mu_\nu$$

where for any two operators, A and B,

$$[A, B] \equiv AB - BA. \qquad (1.2)$$

The subscripts and superscripts take on the values 0, 1, 2, 3. The value zero assigned to μ in x^μ or p_μ will denote the temporal and energy component of the four-vectors x^μ or p_μ respectively; while the remaining values for μ will denote the spatial and momentum components respectively. δ^μ_ν is the well known Kronecker delta

function defined by

$$\delta^{\mu}_{\nu} = 0, \ \mu \neq \nu,$$
$$= 1, \ \mu = \nu. \tag{1.3}$$

We may pass from a contravariant or covariant representation to a covariant or contravariant representation of these vectors through the intermediary of the flat-space metric tensor $\eta^{\mu\nu}$ or $\eta_{\mu\nu}$ defined by

$$\eta_{\mu\nu} = \eta^{\mu\nu} = 1, \ \mu = \nu = 0$$
$$= -1, \ \mu = \nu = 1, 2, 3 \tag{1.4}$$
$$= 0, \ \mu \neq \nu.$$

For if we bear in mind the summation convention of Tensor Analysis

$$p^{\mu} = \eta^{\mu\sigma} p_{\sigma}, \ k_{\mu} = \eta_{\mu\sigma} k^{\sigma}. \tag{1.5}$$

In the same manner any superscript or subscript may be raised or lowered in a tensor expression by suitable multiplication by either the $\eta^{\mu\nu}$ or $\eta_{\mu\nu}$.

2. Matrix Elements of $F(p_\mu, x^\mu)$

In this section we shall tabulate for future reference some matrix elements of various combinations of the operators x^μ and p_μ. In the notation of Dirac[1], the matrix elements of a function of the displacement operator $f(p_\mu)$ in a representation with the p_μ's diagonal are given by

$$\langle p'|f(p_\mu)|p''\rangle = f(p'_\mu)\,\delta(p'-p'') \tag{1.6}$$

where for brevity we write p' and p'' to mean the set p'_0, p'_1, p'_2, p'_3 and $p''_0, p''_1, p''_2, p''_3$, respectively; while $\delta(p'-p'')$ stands for the four-dimensional Dirac function

$$\delta(p'-p'') \equiv \delta(p'_0-p''_0)\,\delta(p'_1-p''_1)\,\delta(p'_2-p''_2)\,\delta(p'_3-p''_3). \tag{1.7}$$

Upon making use of the identity

$$a_\nu \frac{\partial}{\partial a_\mu}\,\delta(a-b) = -\,\delta^\mu_\nu\,\delta(a-b) \tag{1.8}$$

and (1.1) we can write

$$\langle p'|x^\mu|p''\rangle = +i\,\eta^{\mu\nu'}\,\delta_{\nu'}(p'-p''), \tag{1.9}$$

where $\delta_{\nu'}(p'-p'')$ denotes the derivative of $\delta(p'-p'')$ with respect to p'^{ν}. Now if $F(x,p)$ can be written as a double power series in the operators x^{μ} and p_{ν} we can show upon repeatedly applying (1.9), (1.6) and the rules of matrix multiplication,

$$\langle p'|F(p_{\mu},x^{\nu})|p''\rangle = F(p'_{\mu}, i\frac{\partial}{\partial p'_{\nu}})\delta(p'-p''). \qquad (1.10)$$

On the other hand in a representation with the x^{μ} diagonal

$$\langle x'|F(p_{\mu},x^{\nu})|x''\rangle = F(-i\frac{\partial}{\partial x'^{\mu}}, x'^{\nu})\delta(x'-x''). \qquad (1.11)$$

The transformation function which enables one to go from a representation with the p's diagonal to one with the x's diagonal is well known to be

$$\langle x'|p'\rangle = \frac{1}{4\pi^2}\, e^{ip'_{\mu}x'^{\mu}} = \langle p'|x'\rangle^*. \qquad (1.12)$$

(1.12) satisfies

$$\begin{aligned}
&\int \langle x'|p'\rangle d^4p' \langle p'|x''\rangle = \delta(x'-x''),\\
&\int \langle p'|x'\rangle d^4x' \langle x'|p''\rangle = \delta(p'-p'').
\end{aligned} \qquad (1.13)$$

For the special case $F(p,x) = e^{ik_\mu x^\mu}$, with k's being C- numbers, that is, those numbers which commute with all of the operators appearing in the analysis, application of (1.10) yields

$$\langle p' | e^{ik_\mu x^\mu} | p'' \rangle = \delta(p' - p'' - k). \tag{1.14}$$

Similarly,

$$\langle x' | e^{ic_\mu p^\mu} | x'' \rangle = \delta(x' - x'' + c). \tag{1.15}$$

3. <u>Formal Expansion of $F(p,x)$</u>

In some previous work[2] it was shown that there exists a formal expansion of operator functions of the operators p_μ and x^μ which satisfy the commutation relations (1.1) in terms of certain basic operator functions defined by

$$U_{n'k'} \equiv \varphi_{n'} \left(p - \frac{k'}{2}\right) e^{ik'_\mu x^\mu}, \tag{1.16}$$

where

$$\begin{gathered} \int \varphi_{n'}(p') \varphi_{n''}(p') d^4 p' = \delta_{n'n''}, \\ \delta_{n'n''} \equiv \delta_{n_0' n_0''} \delta_{n_1' n_1''} \delta_{n_2' n_2''} \delta_{n_3' n_3''} \end{gathered} \tag{1.17}$$

In terms of these basic functions we could write the formal expansion

$$F(p,x)=\sum_{n'}\int (\mathrm{Tr}\, F(p,x)\tilde{U}_{n'k'})\,U_{n'k'}\,d^4k', \qquad (1.18)$$

where $\tilde{U}_{n'k'}$ is the adjoint of $U_{n'k'}$ and the symbol $(\mathrm{Tr}\, A)$ denotes the trace of the operator A. Consequently, the expansion coefficients are independent of the representation of the operators.

Now, the expansion could be generalized in the following manner by considering our basic functions to be of the form

$$V_{n'k'} \equiv \varphi_{n'}(p-\tfrac{k'}{2})\,\psi_{k'}(x), \qquad (1.19)$$

where we shall require that the ψ's also form a complete orthogonal set satisfying

$$1/(2\pi)^4\int \psi_{k'}(x')\,\psi^*_{k''}(x')\,d^4x' = \delta_{k'k''} \text{ or } \delta(k'-k'') \qquad (1.20)$$

We shall now show that

$$\mathrm{Tr}\,(V_{n'k'}\,\tilde{V}_{n''k''}) = \delta_{n'n''}\,\delta_{k'k''} \qquad (1.21)$$

$$\mathrm{Tr}(V_{n'k'}\tilde{V}_{n''k''})=\mathrm{Tr}\,(\varphi_{n'}(p-\tfrac{k'}{2})\psi_{k'}(x)\psi^*_{k''}(x)\varphi^*_{n''}(p-\tfrac{k''}{2}))$$

$$=\mathrm{Tr}\,(\varphi^*_{n''}(p-\tfrac{k''}{2})\varphi_{n'}(p-\tfrac{k'}{2})\psi_{k'}(x)\psi^*_{k''}(x))$$

upon using the cyclic property of the trace:

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB).$$ **Hence**

$$\mathrm{Tr}(V_{n'k'}\tilde{V}_{n''k''}) = \int \varphi^*_{n''}(p' - \tfrac{k''}{2})\varphi_{n'}(p' - \tfrac{k'}{2})\langle p'|\psi_{k'}(x)\psi^*_{k''}(x)|p'\rangle d^4p'.$$

But,

$$\langle p'|\psi_{k'}(x)\psi^*_{k''}(x)|p'\rangle =$$

$$\int \langle p|x'\rangle\langle x'|\psi_{k'}(x)\psi^*_{k''}(x)|x''\rangle\langle x''|p'\rangle d^4x' d^4x'' =$$

$$\int \langle p'|x'\rangle\langle x'|p'\rangle\psi_{k'}(x')\psi^*_{k''}(x')\, d^4x' =$$

$$1/(2\pi)^4 \int \psi_{k'}(x')\psi^*_{k''}(x')\, d^4x',$$

upon using (1.12). Consequently,

$$\mathrm{Tr}(V_{n'k'}\tilde{V}_{n''k''}) =$$

$$1/(2\pi)^4 \iint \varphi^*_{n''}(p' - \tfrac{k''}{2})\varphi_{n'}(p' - \tfrac{k'}{2})\psi_{k'}(x')\psi_{k'}(x'')d^4p' d^4x',$$

so that upon making use of (1.17) and (1.20), (1.21) follows.

Now if we take as our criterion of approximation to the operator function $F(p,\varkappa)$ the smallness of the number

$$\varepsilon^2 \equiv Tr\Big(F(p,\varkappa)-\sum_{n'k'} a_{n'k'} V_{n'k'}\Big)\Big(\tilde{F}(p,\varkappa)-\sum_{n''k''} a^*_{n''k''} \tilde{V}_{n''k''}\Big) \quad (1.22)$$

it is readily seen that $\varepsilon^2 > 0$ is least if the coefficients $a_{n'k'}$ in the expansion of $F(p,\varkappa)$ satisfy

$$a_{n'k'} = Tr\left(F(p,\varkappa)\,\tilde{V}_{n'k'}\right), \quad (1.23)$$

as we see upon applying (1.21). Thus our expansions satisfy a "least square" requirement if $F(p,\varkappa)$ is approximated by the expansion

$$F(p,\varkappa) = \sum_{n'k'} Tr\left(F(p,\varkappa)\,\tilde{V}_{n'k'}\right) V_{n'k'} \quad (1.24)$$

$\sum_{n'k'}$ denotes summations or integrations over n' and/or k'.

We may further generalise the expansion by noting that the argument of the φ's in (1.19) may be changed to $p - ck'$ where c is any c- number.

II. PROPERTIES OF BASIC FUNCTIONS

4. Representative of the Second Kind (R.S.)

In Quantum Mechanics the representative of a ket $|a\rangle$ is obtained by scalar multiplication by a complete set of basic bras. The set of numbers obtained in this manner is called the representative of the ket $|a\rangle$. The representative of an operator A is obtained by multiplying on the left by a member of the basic bras, say, $\langle\lambda_i|$ and then on the right by the basic ket $|\lambda_j\rangle$. The set $\langle\lambda_i|A|\lambda_j\rangle$ constitutes the representative of the operator A or the matrix elements of A in a representation with λ diagonal. We shall refer to these representatives as representatives of the first kind.

The representative of the second kind (R.S.) of the operator $F(p,x)$ shall be defined as the set of numbers

$$\mathrm{Tr}\,(F\,\widetilde{U}_{n'k'}) \tag{2.1}$$

where $\widetilde{U}_{n'k'}$ is the adjoint of the basic operator $U_{n'k'}$. Knowledge of the set of numbers (2.1) is equivalent to knowledge of the operator F, since we have from (1.18)

$$F(p,x)=\sum_{n'}\int(\mathrm{Tr}\,F\,\widetilde{U}_{n'k'})\,U_{n'k'}\,d^4k'\,, \tag{2.2}$$

in the sense of a "least square" requirement. More generally we may take the R.S. of F relative to the basic functions $V_{m'\ell'}$ defined in (I-3) to be

$$\mathrm{Tr}\,(F\,\widetilde{V}_{m'\ell'}) \tag{2.3}$$

The R.S. of $V_{m''\ell''}$ relative to the basic functions $V_{m'\ell'}$ is

$$\mathrm{Tr}\,(V_{m''\ell''}\widetilde{V}_{m'\ell'}) = \delta_{m'm''}\,\delta_{\ell'\ell''}\,, \tag{2.4}$$

as we see from (1.21) and we note that these R.S.'s are real numbers. In (2.4) and (2.2) the notation implies that the n's are discrete and the k's are continuous. In any stage of the analysis we may consider the n's and the k's to be either continuous or discrete but shall use a notation which implies discreteness for ease in manipulation, it being a simple manner to effect changes in our expressions in any particular case.

Now if A and B are two operator functions of the p's and z's we have

$$\begin{aligned}(\mathrm{Tr}\,(AB))^* &= \int \langle p'|A|p'''\rangle^* d^4p'''\langle p'''|B|p'\rangle^* d^4p' \\ &= \int \langle p'''|\widetilde{A}|p'\rangle\langle p'|\widetilde{B}|p'''\rangle d^4p' d^4p''' \\ &= \mathrm{Tr}\,(\widetilde{B}\widetilde{A}) = \mathrm{Tr}\,(\widetilde{A}\widetilde{B}).\end{aligned} \tag{2.5}$$

In a similar manner we may show that

$$\left(\mathrm{Tr}\,(A\,B\,C)\right)^{*} = \mathrm{Tr}\,(\widetilde{C}\,\widetilde{B}\widetilde{A}) \tag{2.6}$$

In view of (2.5) we have the following relationship between the R.S. of F relative to the basic functions $V_{m'\ell'}$ and the R.S. of $\widetilde{F}$ relative to the basic functions $\widetilde{V}_{m'\ell'}$

$$\mathrm{Tr}\,(F\,\widetilde{V}_{m'\ell'}) = \left(\mathrm{Tr}\,(\widetilde{F}\,V_{m'\ell'})\right)^{*}. \tag{2.7}$$

The equations establishing the connection between the representatives (R.S.) of F and F and the R.S. of $\widetilde{F}$ and $\widetilde{F}$ respectively are

$$\begin{aligned} F &= \sum_{m'\ell'} \mathrm{Tr}\,(F\,\widetilde{V}_{m'\ell'})\,V_{m'\ell'} \\ \widetilde{F} &= \sum_{m'\ell'} \mathrm{Tr}\,(\widetilde{F}\,V_{m'\ell'})\,\widetilde{V}_{m'\ell'} \end{aligned} \tag{2.8}$$

It is seen that the second equation of (2.8) may be obtained by taking the adjoint of both sides of the first equation and applying (2.5).

5. <u>Transformation Function of Second Kind: T.F.S.</u>

If we have two complete sets of basic bras $\langle\lambda'|$ and $\langle\mu'|$,

a ket $|A\rangle$ will have two representatives $\langle\lambda'|A\rangle$ and $\langle\mu'|A\rangle$, so that

$$\begin{aligned}\langle\lambda'|A\rangle &= \sum_{\mu'} \langle\lambda'|\mu'\rangle\langle\mu'|A\rangle,\\ \langle\mu'|A\rangle &= \sum_{\lambda'} \langle\mu'|\lambda'\rangle\langle\lambda'|A\rangle,\end{aligned} \tag{2.9}$$

since

$$\sum_{\mu'} |\mu'\rangle\langle\mu'| = \sum_{\lambda'} |\lambda'\rangle\langle\lambda'| = 1. \tag{2.10}$$

(2.9) shows that either representative of $|A\rangle$ can be expressed in terms of the other representative and the coefficients $\langle\lambda'|\mu'\rangle$ or $\langle\mu'|\lambda'\rangle$. These latter numbers are called transformation functions in Quantum Mechanics.

Now let us consider two sets of basic operators $V_{m'\ell'}$ and $W_{n'k'}$ which satisfy

$$\begin{aligned}Tr(V_{m'\ell'}\widetilde{V}_{m''\ell''}) &= \delta_{m'm''}\,\delta_{\ell'\ell''},\\ Tr(W_{n'k'}\widetilde{W}_{n''k''}) &= \delta_{n'n''}\,\delta_{k'k''}.\end{aligned} \tag{2.11}$$

An operator function F may be expressed in terms of either set as

$$F = \sum_{m'\ell'} \mathrm{Tr}\,(F\,\widetilde{V}_{m'\ell'})\,V_{m'\ell'}\,, \tag{2.12}$$

or

$$F = \sum_{n'k'} \mathrm{Tr}\,(F\,\widetilde{W}_{n'k'})\,W_{n'k'} \tag{2.13}$$

Now if in (2.12) we replace F by $W_{n'k'}$, we obtain

$$W_{n'k'} = \sum_{m'\ell'} \mathrm{Tr}\,(W_{n'k'}\,\widetilde{V}_{m'\ell'})\,V_{m'\ell'}\,. \tag{2.14}$$

Upon putting (2.14) into (2.13) there results

$$F = \sum_{n'k'} \sum_{m'\ell'} \mathrm{Tr}(F\widetilde{W}_{n'k'})\,\mathrm{Tr}\,(W_{n'k'}\widetilde{V}_{m'\ell'})\,V_{m'\ell'}\,. \tag{2.15}$$

Consequently, if we multiply both sides of (2.15) by $\widetilde{V}_{m''\ell''}$ and take the trace, we have using (2.11)

$$\mathrm{Tr}\,(F\widetilde{V}_{m'\ell'}) = \sum_{n'k'} \mathrm{Tr}\,(F\widetilde{W}_{n'k'})\,\mathrm{Tr}\,(W_{n'k'}\widetilde{V}_{m'\ell'})\,. \tag{2.16}$$

The interchange of the W's and V's appearing in (2.16) gives

$$\mathrm{Tr}\,(F\widetilde{W}_{n'k'}) = \sum_{m'\ell'} \mathrm{Tr}\,(F\widetilde{V}_{m'\ell'})\,\mathrm{Tr}\,(V_{m'\ell'}\widetilde{W}_{n'k'})\,. \tag{2.17}$$

(2.16) and (2.17) establishes a relationship between the R.S. of F relative to the set of basic functions $\widetilde{V}_{m'\ell'}$ and the R.S. of F relative to the set of basic functions $W_{n'k'}$ involving the expressions

$$\mathrm{Tr}\,(W_{n'k'}\,\widetilde{V}_{m'\ell'}),\ \mathrm{Tr}\,(V_{m'\ell'}\,\widetilde{W}_{n'k'}). \tag{2.18}$$

The quantities in (2.18) are defined to be transformation functions of the second kind: T.F.S. These transformation functions satisfy

$$\mathrm{Tr}\,(W_{n'k'}\,\widetilde{V}_{m'\ell'}) = \left(\mathrm{Tr}\,(V_{m'\ell'}\,\widetilde{W}_{n'k'})\right)^{*}, \tag{2.19}$$

as we see when we apply (2.5).

These T.F.S.'s are seen to satisfy also

$$\delta_{m'm''}\,\delta_{\ell'\ell''} = \sum_{n'k'} \mathrm{Tr}\,(V_{m'\ell'}\,\widetilde{W}_{n'k'})\,\mathrm{Tr}\,(W_{n'k'}\,\widetilde{V}_{m''\ell''}) \tag{2.20}$$

$$\delta_{n'n''}\,\delta_{k'k''} = \sum_{m'\ell'} \mathrm{Tr}\,(W_{n'k'}\,\widetilde{V}_{m'\ell'})\,\mathrm{Tr}\,(V_{m'\ell'}\,\widetilde{W}_{n''k''}), \tag{2.21}$$

which may be verified by replacing F in (2.16) and (2.17) by $V_{m''\ell''}$ and $W_{n''k''}$ respectively and using (2.11).

6. Matrix Elements of the Second Kind: M.E.S.

Let us consider the expansion of the operator $W_{n'k'}F$ in terms of the set of basic functions $W_{n'k'}$. Upon replacing F in (2.13) by $W_{n'k'}F$ we obtain

$$W_{n'k'}F = \sum_{n''k''} Tr\,(W_{n'k'}F\widetilde{W}_{n''k''})\,W_{n''k''}\,. \tag{2.22}$$

If we now multiply (2.22) on the right by G' and then trace both sides of the equation we obtain

$$Tr\,(W_{n'k'}FG') = \sum_{n''k''} Tr\,(W_{n'k'}F\widetilde{W}_{n''k''})\,Tr\,(W_{n''k''}G'), \tag{2.23}$$

where G' is another operator. If G' is now replaced by $G\widetilde{W}_{n'''k'''}$, (2.23) yields

$$Tr\,(W_{n'k'}FG\widetilde{W}_{n'''k'''}) =$$

$$\sum_{n''k''} Tr\,(W_{n'k'}F\widetilde{W}_{n''k''})\,Tr\,(W_{n''k''}G\widetilde{W}_{n'''k'''}), \tag{2.24}$$

which is analogous to the law of matrix multiplication in Quantum Mechanics

$$\langle\lambda'|FG|\lambda''\rangle = \sum_{\lambda'''}\langle\lambda'|F|\lambda'''\rangle\langle\lambda'''|G|\lambda''\rangle \tag{2.25}$$

As a result we shall define the <u>matrix elements of the second kind: (M.E.S.) of an operator, F, say, relative to the set of basic functions $W_{n'k'}$</u> to be

$$\mathrm{Tr}\,(W_{n'k'}\,F\,\widetilde{W}_{n''k''}) \tag{2.26}$$

From (2.3) we see that the M.E.S. of F is merely the R.S. of the operator $W_{n'k'}\,F$ relative to the set of W's or the R.S. of $F\,\widetilde{W}_{n''k''}$ relative to the set of $\widetilde{W}$'s. The adjoint $\widetilde{F}$ of F has for its M.E.S.

$$\mathrm{Tr}\,(W_{n'k'}\,\widetilde{F}\,\widetilde{W}_{n''k''}), \tag{2.27}$$

so that from (2.6)

$$\mathrm{Tr}\,(W_{n'k'}\,\widetilde{F}\,\widetilde{W}_{n''k''}) = \left(\mathrm{Tr}\,(W_{n''k''}\,F\,\widetilde{W}_{n'k'})\right)^{*} \tag{2.28}$$

which is the conjugate complex of the <u>transposed</u> M.E.S. of F.
If F is hermitian then

$$\mathrm{Tr}\,(W_{n'k'}\,F\,\widetilde{W}_{n''k''}) = \mathrm{Tr}\,(W_{n'k'}\,\widetilde{F}\,\widetilde{W}_{n''k''})$$

$$= \left(\mathrm{Tr}\,(W_{n''k''}\,F\,\widetilde{W}_{n'k'})\right)^{*} \tag{2.29}$$

which is analogous to the property of the matrix elements of a hermitian operator F

$$\langle p' | F | p'' \rangle = \langle p'' | F | p' \rangle^* \tag{2.30}$$

The M.E.S. of F relative to the V's may be obtained from the M.E.S. of F relative to the W's through the intermediary of the T.F.S.'s defined in (2.18):

$$\mathrm{Tr}\,(V_{m'\ell'}\, F\, \widetilde{V}_{m''\ell''}) =$$

$$\sum_{n'k',n''k''} \mathrm{Tr}\,(V_{m'\ell'}\, \widetilde{W}_{n'k'})\,\mathrm{Tr}(W_{n'k'}\, F\, \widetilde{W}_{n''k''})\,\mathrm{Tr}(W_{n''k''}\, \widetilde{V}_{m''\ell''}) \tag{2.31}$$

(2.31) may be verified by noting that (2.16) implies

$$\sum_{n'k'} \mathrm{Tr}(V_{m'\ell'}\, \widetilde{W}_{n'k'})\,\mathrm{Tr}(W_{n'k'}\, F\, \widetilde{W}_{n''k''}) = \mathrm{Tr}(V_{m'\ell'}\, F\, \widetilde{W}_{n''k''}), \tag{2.32}$$

and that

$$\sum_{n''k''} \mathrm{Tr}(V_{m'\ell'}\, F\, \widetilde{W}_{n''k''})\,\mathrm{Tr}(W_{n''k''}\, \widetilde{V}_{m''\ell''}) = \mathrm{Tr}\,(V_{m'\ell'}\, F\, \widetilde{V}_{m''\ell''}), \tag{2.33}$$

which follows from (2.17) upon interchanging the role played by W and V therein.

We must note that knowledge of the M.E.S. of F enables us to infer either $W_{m'l'}F$ or $F\widetilde{W}_{n'l'}$ so that if $W_{m'l'}$ possesses an inverse then F is accordingly determined. The M.E.S. of F is a function of four sets of numbers while the R.S. of F is a function of two sets. Consequently in principle it would be simpler to represent F via the R.S. of F rather than through the M.E.S. of F.

However, we may utilize knowledge of the M.E.S. of F to determine F by means of the expression

$$F = \sum_{n'k', n''k''} Tr(\widetilde{W}_{n'k'})\, Tr(W_{n'k'} F \widetilde{W}_{n''k''}) W_{n''k''} . \tag{2.34}$$

(2.34) follows by noting that

$$W_{n'k'} F = \sum_{n''k''} Tr(W_{n'k'} F \widetilde{W}_{n''k''}) W_{n''k''} , \tag{2.35}$$

by (2.22) and that

$$\sum_{n'k'} Tr(\widetilde{W}_{n'k'}) W_{n'k'} = 1 , \tag{2.36}$$

which is obtained by putting $F = 1$ in (2.12).

7. The Basic Functions $1/(2\pi)^2\, e^{in'^\mu(p_\mu - ak'_\mu/2)} e^{ik'_\mu x^\mu}$

Let us define the basic functions $X_{n'k'}$ as

$$X_{n'k'} = 1/(2\pi)^2\, e^{in'^\mu(p_\mu - ak'_\mu/2)}\, e^{ik'_\mu x^\mu} \tag{2.37}$$

where a is any c- number. This set satisfies

$$\mathrm{Tr}\,(X_{n'k'}\,\widetilde{X}_{n''k''}) = \delta(n'-n'')\,\delta(k'-k'') \tag{2.37a}$$

where $\delta(n'-n'')$ and $\delta(k'-k'')$ are four-dimensional Dirac functions defined similarly to $\delta(p'-p'')$ in (1.7). We may verify (2.37a) in the following manner:

$$X_{n'k'}\,\widetilde{X}_{n''k''}$$

$$= 1/16\pi^4\, e^{in'^\mu(p_\mu - ak'_\mu/2)}\, e^{i(k'_\mu - k''_\mu)x^\mu} \times$$

$$e^{-in''^\mu(p_\mu - ak''_\mu/2)}, \tag{2.38}$$

which becomes upon making use of

$$e^{ik_\mu x^\mu} f(p,x)\, e^{-ik_\mu x^\mu} = f(p-k, x), \tag{2.39}$$

$$X_{n'k'}\tilde{X}_{n''k''} = \frac{1}{16\pi^4} e^{-i(a n'^{\mu} k'_{\mu}/2 - a n''^{\mu} k''_{\mu}/2 - n''^{\mu} k'_{\mu} + n'^{\mu} k''_{\mu})} \times$$

$$e^{i(n'^{\mu} - n''^{\mu})p_{\mu}} e^{i(k'_{\mu} - k''_{\mu})x^{\mu}} . \tag{2.40}$$

Consequently, (2.37) follows because

$$\mathrm{Tr}\, e^{ic^{\mu}p_{\mu}} e^{ik_{\mu}x^{\mu}} =$$

$$\int \langle p' | e^{ic^{\mu}p_{\mu}} | p'' \rangle d^4p'' \langle p'' | e^{ik_{\mu}x^{\mu}} | p' \rangle d^4p' =$$

$$\int e^{ic^{\mu}p'_{\mu}} \delta(-k) d^4p' = 16\pi^4 \delta(c)\, \delta(k)$$

from (1.14),

$$\int e^{ic^{\mu}p_{\mu}} d^4p'_{\mu} = 16\pi^4 \delta(c), \tag{2.41}$$

and $\delta(k) = \delta(-k)$.

We could have obtained (2.37) by making use of the expressions obtained in section (I-3). It would have been merely necessary to make the identification

$$\varphi_{n'}(p - c k'/2) \rightarrow 1/(2\pi)^2\, e^{i n'^{\mu}(p_{\mu} - a k'_{\mu}/2)},$$

$$\psi_{k'}(x) \rightarrow e^{i k'_{\mu} x^{\mu}} \tag{2.42}$$

It is seen that this identification assures us that the orthogonality conditions set forth in (1.17) and (1.20) are satisfied. However, the procedure we have followed serves merely for verification of the more general results of (I-3) and to indicate explicitly the manipulations in obtaining the results. It is to be noted that the set of χ's when used for expanding $F(p, x)$ is a natural extension of a Fourier Integral of C- number functions.

III. LINEAR OPERATOR EQUATIONS

8. Matrix Elements of $A(p,x)UB(p,x)$

In order to expedite our considerations of linear operator equations we shall have need for explicit expressions for the matrix elements of $A(p,x)UB(p,x)$ in terms of the matrix elements of U. Now if k_μ is a C- number we have

$$\langle p'|e^{ik_\mu x^\mu}U|p''\rangle = \int \langle p'|e^{ik_\mu x^\mu}|p'''\rangle d^4p'''\langle p'''|U|p''\rangle, \tag{3.1}$$

which becomes upon using (1.14)

$$\begin{aligned}\langle p'|e^{ik_\mu x^\mu}U|p''\rangle &= \int \delta(p'-p'''-k)d^4p'''\langle p'''|U|p''\rangle \\ &= \langle p'-k|U|p''\rangle.\end{aligned} \tag{3.2}$$

If we differentiate both sides of (3.2) partially with respect to k_μ we obtain

$$\begin{aligned}i\langle p'|x^\mu e^{ik_\mu x^\mu}U|p''\rangle &= \frac{\partial}{\partial k_\mu}\langle p'-k|U|p''\rangle \\ &= \frac{-\partial}{\partial p'_\mu}\langle p'-k|U|p''\rangle\end{aligned} \tag{3.3}$$

If we now set $k_\mu = 0$, (3.3) reduces to

$$\langle p' | x^\mu U | p'' \rangle = i \frac{\partial}{\partial p'_\mu} \langle p' | U | p'' \rangle. \tag{3.4}$$

In a similar manner we may show that

$$\langle p' | U x^\mu | p'' \rangle = -i \frac{\partial}{\partial p''_\mu} \langle p' | U | p'' \rangle. \tag{3.5}$$

Now,

$$\langle p' | x^\mu p_\nu U | p'' \rangle = i \frac{\partial}{\partial p'_\mu} \langle p' | p_\nu U | p'' \rangle,$$

$$= i \frac{\partial}{\partial p'_\mu} p'_\nu \langle p' | U | p'' \rangle \tag{3.6}$$

as we see from (3.4) upon replacing U there by $p_\nu U$ and noting that $\langle p' | p_\nu = p'_\nu \langle p' |$. Repeated application of (3.6) suggests that if $A(p, x)$ can be expressed as a power series involving the p's and the x's.

$$\langle p' | A(p,x) U | p'' \rangle = A(p', i \frac{\partial}{\partial p'}) \langle p' | U | p'' \rangle. \tag{3.7}$$

If we consider $\langle p' | U x^\mu p_\nu | p'' \rangle$.

$$\langle p'|U x^\mu p_\nu|p''\rangle = \langle p'|U x^\mu|p''\rangle p''_\nu$$

$$= p''_\nu \left(-i\frac{\partial}{\partial p''_\mu}\right)\langle p'|U|p''\rangle, \qquad (3.8)$$

when we apply $p_\nu|p''\rangle = p''_\nu\,|p''\rangle$ and (3.5). (3.8) also suggests that if $B(p,x)$ can be expressed as a power series involving the p's and x's then

$$\langle p'|U B(p,x)|p''\rangle = \widetilde{B}^*\left(p'', -i\frac{\partial}{\partial p''}\right)\langle p'|U|p''\rangle, \qquad (3.9)$$

where $\widetilde{B}^*(p,x)$ is the complex conjugate of the adjoint of $B(p,x)$. For example if $B(p,x) = i\,x\,p + p\,x$, $\widetilde{B}^*(p,x) = +i\,p\,x + x\,p$, since x and p are hermitian. $\widetilde{B}^*\left(p'' - i\frac{\partial}{\partial p''}\right)$ for this case would become the operator

$$i\,p''\left(-i\frac{\partial}{\partial p''}\right) + \left(-i\frac{\partial}{\partial p''}\right)p''.$$

Consequently,

$$\langle p'|A(p,x)\,U\,B(p,x)|p''\rangle =$$

$$A\left(p', i\frac{\partial}{\partial p'}\right)\widetilde{B}^*\left(p'', -i\frac{\partial}{\partial p''}\right)\langle p'|U|p''\rangle, \qquad (3.10)$$

upon applying (3.7) first with U replaced by $UB(p,\varkappa)$ and then applying (3.9) for the latter operator. Thus if the operators A and B are known (3.10) enables us to express the matrix elements as a product of two differential operators acting on $\langle p'|U|p''\rangle$.

In a similar manner we can show that in a representation with the $\varkappa$'s diagonal

$$\langle \varkappa'|A(p,\varkappa)UB(p,\varkappa)|\varkappa''\rangle =$$

$$A(-i\frac{\partial}{\partial \varkappa'},\varkappa')\widetilde{B}^*(i\frac{\partial}{\partial \varkappa''},\varkappa'')\langle \varkappa'|U|\varkappa''\rangle \tag{3.11}$$

(3.11) could also be obtained by making use of the transformation functions $\langle \varkappa'|p'\rangle$ and $\langle p'|\varkappa'\rangle$ defined in (1.12).

We could also note here that

$$\begin{aligned} \widetilde{B}^*(p'',-i\frac{\partial}{\partial p''}) &= [\widetilde{B}(p'',i\frac{\partial}{\partial p''})]^*, \\ \widetilde{B}^*(i\frac{\partial}{\partial \varkappa''},\varkappa'') &= [\widetilde{B}(-i\frac{\partial}{\partial \varkappa''},\varkappa'')]^*, \end{aligned} \tag{3.11a}$$

since the asterisk in $\widetilde{B}^*$ was introduced to neutralize the appearance of the asterisk acting on C- numbers in forming $\widetilde{B}(p,\varkappa)$ in order to be consistent with our work leading to (3.9).

9. General Linear Operator Equation

The most general way in which the operator U may occur linearly is in an equation of the type

$$\sum_i A_i U B_i = C, \tag{3.12}$$

where the A's, B's and C are presumed to be known operator functions of the p's and the x's. We shall call (3.12) a linear inhomogeneous operator equation for the unknown operator U. If we take the matrix elements of both sides of (3.12) in a representation with the p's diagonal and apply (3.10) to each of the expressions $A_i U B_i$ in the summand of (3.12) we obtain

$$\sum_i A_i(p', i\tfrac{\partial}{\partial p'})\tilde{B}_i^*(p'', -i\tfrac{\partial}{\partial p''})\langle p'|U|p''\rangle = \langle p'|C|p''\rangle \tag{3.12a}$$

On the other hand if we take the matrix elements of (3.12) in a representation with the x's diagonal we obtain upon using (3.11)

$$\sum_i A_i(-i\tfrac{\partial}{\partial x'}, x')\tilde{B}_i^*(i\tfrac{\partial}{\partial x''}, x'')\langle x'|U|x''\rangle = \langle x'|C|x''\rangle. \tag{3.13}$$

(3.12) ad (3.13) are partial differential equations for the matrix elements $\langle p'|U|p''\rangle$ and $\langle \varkappa'|U|\varkappa''\rangle$ respectively. Consequently, if we can solve (3.12a) or (3.13) for the matrix elements the operator U is determined. For if we know the matrix elements of U we have

$$Tr(U\widetilde{W}_{n'k'}) = \iint \langle p'|U|p''\rangle\langle p''|\widetilde{W}_{n'k'}|p'\rangle d^4p'd^4p'' \quad (3.14)$$

which is the R.S. of U relative to the set of basic functions satisfying (2.11). But upon replacing F by U in (2.13) we have

$$U = \sum_{n'k'} Tr\,(U\widetilde{W}_{n'k'})\,W_{n'k'}\,, \quad (3.15)$$

$$U = \sum_{n'k'} \iint \langle p'|U|p''\rangle\langle p''|\widetilde{W}_{n'k'}|p'\rangle W_{n'k'}\,d^4p'd^4p''\,, \quad (3.16)$$

upon substitution of (3.14) in (3.15). In precisely the same way we have in terms of the matrix elements of U in a representation with the $\varkappa$'S diagonal,

$$U = \sum_{n'k'} \iint \langle \varkappa'|U|\varkappa''\rangle\langle \varkappa''|\widetilde{W}_{n'k'}|\varkappa'\rangle W_{n'k'}\,d^4\varkappa'd^4\varkappa''. \quad (3.17)$$

In principle, then, we have reduced the solution of the linear operator equation (3.12) to the solution of a linear inhomogeneous

partial differential equation involving eight independent variables, namely, the four p''s plus the four p'''s or the four x''s plus the four x'''s depending upon whether (3.12) or (3.13) is taken under consideration. (3.16) and (3.17) furnish us explicit expressions for the operator U in terms of the eight operators x^μ and p_μ. It is at once apparent that the solutions (3.16) or (3.17) are independent of the representations for the operators x^μ and p_μ due to the circumstance that the trace of an operator is independent of the representation. Moreover, the solutions (3.16) or (3.17) do not depend essentially upon the nature of the basic functions just so long as they form a complete set satisfying orthogonality conditions of the form exemplified by equations of the type (2.11).

10. Solution Utilizing the Basic Functions $X_{n'k'}$

If in (3.16) we replace the basic functions $W_{n'k'}$ by the basic functions $X_{n'k'}$ we find that we shall have to evaluate the matrix elements $\langle p''|\tilde{X}_{n'k'}|p'\rangle$. Now from (2.37)

$$X_{n'k'} = 1/(2\pi)^2\, e^{in'^\mu(p_\mu - ak'_\mu/2)}\, e^{ik'_\mu x^\mu},$$

$$\tilde{X}_{n'k'} = 1/(2\pi)^2\, e^{-ik'_\mu x^\mu}\, e^{-in'^\mu(p_\mu - ak'_\mu/2)}, \qquad (3.18)$$

so that

$$\langle p''|\tilde{X}_{n'k'}|p'\rangle = 1/(2\pi)^2\,\delta(p''-p'+k')\,e^{-in'^{\mu}(p'_{\mu}-ak'_{\mu}/2)} \tag{3.19}$$

as we see upon applying (1.14) and the laws of matrix multiplication. Consequently

$$\langle p''|\tilde{X}_{n'k'}|p'\rangle X_{n'k'} =$$

$$1/(2\pi)^4\,\delta(p''-p'+k')\,e^{in'^{\mu}(q_{\mu}-p'_{\mu})}e^{ik'_{\mu}x^{\mu}} \tag{3.20}$$

(3.20) is independent of the c- number a.

But

$$U = \sum_{n'k'}\iint\langle p'|U|p''\rangle\langle p''|\tilde{X}_{n'k'}|p'\rangle X_{n'k'}\,d^4p'\,d^4p' \tag{3.21}$$

Upon substituting (3.20) in (3.21) we have

$$U = \sum_{n'k'}\iint\langle p'|U|p''\rangle\,(1/(2\pi)^4)\,\times$$

$$\delta(p''-p'+k')\,e^{in'^{\mu}(q_{\mu}-p'_{\mu})}e^{ik'_{\mu}x^{\mu}}\,d^4p'\,d^4p'' \tag{3.22}$$

or

$$U = \sum_{n'k'}\iint\langle p'|U|p'-k'\rangle\,(1/(2\pi)^4)\,e^{in'^{\mu}(p_{\mu}-p'_{\mu})}e^{ik'_{\mu}x^{\mu}}\,d^4p' \tag{3.23}$$

But, recalling that $\sum_{n'k'}$ denotes either integration or summation we have if n' and k' are continuous so that

$$\sum_{n'k'} (\ \ ) \longrightarrow \iint (\ \ )\, d^4n'\, d^4k'$$

$$\int e^{in'^{\mu}(p_{\mu}-p'_{\mu})}\, d^4n' = (2\pi)^4\, \delta\,(p-p'). \qquad (3.24)$$

(3.24) is formal in character since we are treating p as a C-number when actually it is an operator. Nevertheless, if we take the matrix elements of the left side of (3.24) in a representation with the p's diagonal and then integrate we see that the operator $\delta\,(p-p')$ has matrix elements $\langle p''|\delta\,(p-p')|p'''\rangle = \delta\,(p''-p')\,\delta\,(p''-p''')$ which is consistent with $\langle p'|f\,(p)|p''\rangle = f\,(p')\,\delta\,(p'-p'')$. Consequently, (3.23) becomes

$$U = \iint \langle p'|U|p'-k'\rangle\, \delta\,(p-p')\, e^{ik'_{\mu}\nu^{\mu}}\, d^4p'\, d^4k'. \qquad (3.25)$$

Now here again even though p is an operator we may show that it is not inconsistent to make use of the familiar property of the δ-function

$$\int f\,(p')\,\delta\,(p'-a)\, d^4p' = f(a), \qquad (3.26)$$

so that (3.25) becomes

$$U = \int \langle p | U | p - k' \rangle e^{i k'_\mu x^\mu} d^4 k' . \tag{3.27}$$

We must emphasize that $\langle p | U | p - k' \rangle$ is now to be considered a function of the operators p_μ with matrix elements

$$\langle p' | \langle p | U | p - k' \rangle | p'' \rangle = \langle p' | U | p' - k' \rangle \delta (p' - p'') . \tag{3.28}$$

$\langle p | U | p - k' \rangle$ is obtained from $\langle p' | U | p'' \rangle$ by the simple procedure of replacing p' and p'' respectively by p and $p - k'$. It is readily verified that the trace of the left side of (3.27) is identical to the trace of the right hand side, and this provides an additional check on our assertions regarding the consistency of our manipulations.

Examination of (3.27) reveals the striking resemblance of the expansion to the familiar expansion of the $C-$ number Fourier integral expansion and thus would provide a suitable starting point in studying departures from $C-$ number fields.

It would appear from the structure of our basic functions $\chi_{n' k'}$ that we could obtain a similar expression equivalent to (3.27) with the roles of x^μ and p_μ interchanged. To show that such is the case we would have to start with (3.17) instead of (3.16)

and evaluate our matrix elements in a representation with the x's diagonal. In place of (3.19) we will have

$$\langle x''|\widetilde{X}_{n'k'}|x'\rangle = 1/(2\pi)^2 \, e^{-ik'_\mu(x''^\mu - an'^\mu/2)} \, \delta(x''-x'-n'), \quad (3.29)$$

so that

$$\langle x''|\widetilde{X}_{n'k'}|x'\rangle X_{n'k'} =$$

$$1/(2\pi)^4 \, \delta(x''-x'-n') \, e^{in'_\mu p^\mu} \, e^{ik'_\mu(x'^\mu - x''^\mu)} \quad (3.30)$$

But U can also be written as

$$U = \sum_{n'k'} \iint \langle x'|U|x''\rangle \langle x''|\widetilde{X}_{n'k'}|x'\rangle X_{n'k'} \, d^4x' \, d^4x'', \quad (3.31)$$

upon replacing the basic functions $W_{n'k'}$ by the basic functions $X_{n'k'}$ in (3.17). If we introduce (3.30) in (3.31) we obtain

$$U = \sum_{n'k'} \iint 1/(2\pi)^4 \langle x'|U|x''\rangle \delta(x''-x'-n') \, e^{in'_\mu p^\mu} \, e^{ik'_\mu(x'^\mu - x''^\mu)} \, d^4x' \, d^4x''. \quad (3.32)$$

Again recalling our convention regarding $\sum_{n'k'}$ to denote integration or summation we have upon using the integration interpretation of $\sum_{n'k'}$

$$U = \int e^{in'^{\mu} p_{\mu}} \langle x - n' | U | x \rangle \, d^4 n', \tag{3.33}$$

in much the same manner that (3.27) was obtained. Here also we must point out that $\langle x - n' | U | x \rangle$ is an operator function of x and is obtained from the C- number $\langle x' | U | x'' \rangle$ by replacing x' by $x - n'$ and x'' by x. We should also note that although the exponential part in the integrand of (3.27) appears on the right the exponential part of the integrand of (3.33) appears on the left hand side. This is so because the p's do not commute with the x's. We may however write (3.33) with the exponent on the right in the integrand by making use of

$$e^{in'^{\mu} p_{\mu}} f(p, x) e^{-in'^{\mu} p_{\mu}} = f(p, x + n'), \tag{3.34}$$

so that we have also

$$U = \int \langle x | U | x + n' \rangle e^{in'^{\mu} p_{\mu}} \, d^4 n'. \tag{3.35}$$

Similarly, by making use of (2.39), (3.27) may be written as

$$\begin{aligned} U &= \int \langle p | U | p - k' \rangle e^{ik'_{\mu} x^{\mu}} \, d^4 k' \\ &= \int e^{ik'_{\mu} x^{\mu}} \langle p + k' | U | p \rangle \, d^4 k'. \end{aligned} \tag{3.36}$$

The expressions (3.35), (3.33), (3.27) and (3.36) are quite general and are independent of whether U is a solution of the linear operator equation (3.12) or not. Aside from this, the work that we have done in this section indicates in a simple manner the role played by the matrix elements of U satisfying (3.12a) as being essentially of the nature of Fourier coefficients which in general however are operator functions of p or $\varkappa$ depending upon whether the expansion (3.27) and (3.36) or (3.33) or (3.35) is used.

11. The Operator Equation $AS - SB = 0$

It is of interest in connection with our work on operator equations to give consideration to the operator equation

$$AS - SB = 0. \tag{3.37}$$

If S possesses an inverse S^{-1} then (3.47) can be said to arise from a similarity transformation

$$A = SBS^{-1}, \tag{3.38}$$

so that the solution of (3.37) for the operator S can be connected with the problem of finding the operator S which defines a

similarity transformation if the original operator B and the operator A resulting from such a transformation are known. Moreover, if the operator S is unitary: $S^{-1} = \tilde{S}$, then (3.38) is a unitary transformation and the problem is equivalent to finding the unitary operator S if the operator B and the operator A resulting from the unitary transformation are given explicitly.

If we restrict ourselves to the case with the operators A and B functions of the operators p_μ and x^μ satisfying the commutation relationships given by (1.1) we have from (3.12a) the equivalent partial differential equation for the matrix elements of S in a representation with the p's diagonal

$$[A(p', i\frac{\partial}{\partial p'}) - \tilde{B}^*(p'', -i\frac{\partial}{\partial p''})]\langle p'|S|p''\rangle = 0. \qquad (3.39)$$

(3.39) is separable in the sets of variables p' and p'' so that we could assume a solution for the matrix elements of S to be of the form

$$\langle p'|S|p''\rangle = P'(p')\,P''(p''), \qquad (3.40)$$

where as indicated P' is a function of p' alone and P'' a function of p'' alone. Consequently, (3.39) is equivalent to the two partial differential equations

$$[A(p', i\frac{\partial}{\partial p'}) - \lambda]\, P' = 0, \tag{3.41}$$

and

$$[\widetilde{B}^*(p'', -i\frac{\partial}{\partial p''}) - \lambda]\, P'' = 0, \tag{3.42}$$

for the functions P' and P'' respectively. λ is a C- number which we can now use to label the P's. The general solution of (3.39) is

$$\langle p' | S | p'' \rangle = \int a(\lambda) P'_\lambda\, P''_\lambda\, d\lambda, \tag{3.43}$$

where $a(\lambda)$ is an arbitrary function of λ. It would appear then that there exists an infinite number of operators S satisfying (3.37).

Example:-

As an example let us consider the operator equation (3.37) with

$$A = x_\mu, \; B = p_\mu. \tag{3.44}$$

(3.39) becomes

$$\left(i\frac{\partial}{\partial p'^{\mu}} - p''_{\mu}\right)\langle p'|S|p''\rangle = 0. \tag{3.45}$$

Consequently,

$$\langle p'|S|p''\rangle = \varphi(p''; p'_{\nu\neq\mu})\, e^{-ip'^{\mu}p''_{\mu}}, \tag{3.46}$$

where φ is an arbitrary function of the indicated arguments and the summation convention is being used in the exponential. For simplicity we may choose this arbitrary function to be $(2\pi)^{-2}$ so that denoting S corresponding to this choice by S_0, (3.46) becomes

$$\langle p'|S_0|p''\rangle = (2\pi)^{-2}\, e^{-ip'^{\mu}p''_{\mu}}. \tag{3.47}$$

Now

$$\langle p'|\tilde{S}_0|p''\rangle = (2\pi)^{-2}\, e^{ip'^{\mu}p''_{\mu}}. \tag{3.48}$$

Hence

$$\begin{aligned}\langle p'|S_0\tilde{S}_0|p''\rangle &= (2\pi)^{-4}\int e^{-ip'^{\mu}p'''_{\mu}}\, e^{ip'''^{\mu}p''_{\mu}}\, d^4p''' \\ &= \delta(p'-p''),\end{aligned} \tag{3.49}$$

which implies that

$$S_o \tilde{S}_o = 1 . \tag{3.50}$$

Similarly we may show that

$$\tilde{S}_o S_o = 1. \tag{3.51}$$

But if we interchange the rôles played by A and B in (3.37) we obtain assuming that S possesses an inverse S^{-1}, (3.45) with the primed and twice primed expressions interchanged and i replaced by $-i$ so that we can write

$$\langle p' | S_o^{-1} | p'' \rangle = (2\pi)^{-2} e^{i p'^{\mu} p''_{\mu}} , \tag{3.52}$$

$$\langle p' | \tilde{S}_o^{-1} | p'' \rangle = (2\pi)^{-2} e^{-i p'^{\mu} p''_{\mu}} . \tag{3.53}$$

Comparing with (3.47) and (3.48) we conclude

$$S_o^{-1} = \tilde{S}_o ; \; \tilde{S}_o^{-1} = S_o , \tag{3.54}$$

so that in view of (3.50) and (3.51) the operator S_o is unitary. Hence we may write

$$x_\mu = S_0 \, p_\mu \, S_0^{-1} ,$$
$$p_\mu = S_0^{-1} \, x_\mu \, S_0 . \tag{3.55}$$

Now let us compute $S_0 \, x_\mu \, S_0^{-1}$.

$$\langle p' | S_0 \, e^{ik_\mu x^\mu} S_0^{-1} | p'' \rangle =$$

$$\int \langle p' | S_0 | p''' \rangle \langle p''' | e^{ik_\mu x^\mu} | p^{iv} \rangle \langle p^{iv} | \widetilde{S}_0 | p'' \rangle d^4 p''' d^4 p^{iv} .$$

But from (1.14), (3.47) and (3.48) the above may be written as

$$\langle p' | S_0 \, e^{ik_\mu x^\mu} S_0^{-1} | p'' \rangle =$$

$$(2\pi)^{-4} \int e^{-ip'^\mu p'''_\mu} \delta(p''' - p^{iv} - k) e^{ip^{iv\mu} p''_\mu} d^4 p''' d^4 p^{iv} =$$

$$\delta(p' - p'') \, e^{-ik_\mu p_\mu} ,$$

which implies

$$S_0 \, e^{ik_\mu x^\mu} S_0^{-1} = e^{-ik_\mu p^\mu} , \tag{3.56}$$

so that

$$S_0 \, x_\mu \, S_0^{-1} = -p_\mu \, . \tag{3.57}$$

If in (3.44) we replace p_μ by $\alpha^{-1} p_\mu$ and proceed in the same manner we find that (3.47) becomes

$$\langle p' | S_\alpha | p'' \rangle = (2\pi\alpha)^{-2} e^{-i p'^\mu p''_\mu / \alpha} \, ,$$
$$\langle p' | \tilde{S}_\alpha | p'' \rangle = (2\pi\alpha)^{-2} e^{i p'^\mu p''_\mu / \alpha} \, . \tag{3.58}$$

Moreover, we may show that S_α is unitary. Consequently,

$$x_\mu = \alpha^{-1} S_\alpha \, p_\mu \, S_\alpha^{-1} , \tag{3.59}$$

and similarly to (3.57),

$$S_\alpha \, x_\mu \, S_\alpha^{-1} = -\alpha^{-1} p_\mu \, . \tag{3.60}$$

In a straightforward manner we may also show that the matrix elements of S_α^2 are

$$\langle p' | S_\alpha^2 | p'' \rangle = \delta (p' + p'') \tag{3.61}$$

so that the square of the unitary operator S_α has the same matrix elements as the reflection operator. We may also show the S_α^2 anticommutes with p_μ and x^μ.

12. Symbolic Formalism. Transformation Operator.

If A and B are operators the transformation operator $T(A,B)$ is defined by

$$T(A,B)X \equiv AXB, \tag{3.62}$$

where X is another operator. Now

$$\begin{aligned} T(A,B)T(A,B)X &= T(A,B)AXB = T(A^2,B^2)X \\ &= A^2XB \end{aligned} \tag{3.63}$$

(3.63) suggests

$$T^m(A,B)T^n(A,B) = T^{m+n}(A,B) = T(A^{m+n},B^{m+n}) \tag{3.64}$$

for positive integer m and n. Moreover, if A and B possess inverses A^{-1} and B^{-1} respectively we may consider (3.64) to hold for m and n any positive or negative integers,

If A, B, C, and D are operators, then the product of the two transformation operators $T(A,B)$ and $T(C,D)$ acting on an operator X is

$$T(A,B)T(C,D)X = T(A,B)CXD$$

$$= ACXDB$$

$$= T(AC,DB)X. \tag{3.65}$$

Consequently,

$$T(A,B)T(C,D) = T(AC,DB). \tag{3.66}$$

If we have a third transformation operator $T(E,F)$ we have

$$T(A,B)\big(T(C,D)T(E,F)\big) =$$

$$T(A,B)T(CE,FD) = T(ACE,FDB). \tag{3.67}$$

But

$$(T(A,B)T(C,D))T(E,F) =$$

$$T(AC,DB)T(E,F) = T(ACE,FDB), \qquad (3.68)$$

which is the same as the right hand side of (3.67). Thus the transformation operators satisfy the associative law of multiplication.

In general $T(A,B)$ does not commute with $T(C,D)$ unless $[A,C]=0$; $[B,D]=0$. No simple addition laws seem to exist for the sum of two transformation operators. However, we may note

$$T(A+B,C+D) = T(A,C)+T(A,D)+T(B,C)+T(B,D). \qquad (3.69)$$

Now if a and b are any two operators at least one of which possesses an inverse, say a, we have from a formal identity due to Feynman

$$(a+b)^{-1} = a^{-1} - a^{-1}ba^{-1} + a^{-1}ba^{-1}ba^{-1} - \cdots \qquad (3.70)$$

If we replace a and b by $T(A,B)$ and $T(C,D)$ respectively (3.70) becomes

$$[T(A,B)+T(C,D)]^{-1} =$$

$$T^{-1}(A,B)-T^{-1}(A,B)T(C,D)T^{-1}(A,B)$$

$$+T^{-1}(A,B)T(C,D)T^{-1}(A,B)T(C,D)T^{-1}(A,B). \qquad (3.71)$$

Upon making use of (3.66) and assuming that $T(A,B)$ possesses the inverse $T(A^{-1},B^{-1})$, (3.71) may be written as

$$[T(A,B)+T(C,D)]^{-1} = T(A^{-1},B^{-1})-T(A^{-1}CA^{-1},B^{-1}DB^{-1})$$

$$+T(A^{-1}CA^{-1}CA^{-1},B^{-1}DB^{-1}DB^{-1})-\cdots . \qquad (3.72)$$

(3.72) may be used to effect a formal solution of the operator equation

$$AUB+CUD=E, \qquad (3.73)$$

for the unknown operator U, since (3.73) can be written as

$$[T(A,B)+T(C,D)]U=E, \qquad (3.74)$$

so that

$$U = [T(A,B) + T(C,D)]^{-1} E$$

$$= A^{-1} E B^{-1} - (A^{-1} C A^{-1} E B^{-1} D B^{-1})$$

$$+ (A^{-1} C A^{-1} C A^{-1} E B^{-1} D B^{-1} D B^{-1}) - \cdots \quad . \tag{3.75}$$

The U so obtained is clearly not the general solution of (3.73). To obtain the general solution we need to add the solution of the homogeneous equation $AUB+CUD=0$. The usefulness of this method of course will depend upon whether the series (3.75) converges or not.

IV. AN AVERAGING PROCESS DEFINED FOR OPERATOR FIELDS

13. Yukawa Variables

If we consider the matrix elements of an operator U in a representation with x diagonal, say $\langle x'|U|x''\rangle$, we note that they depend upon eight sets of numbers comprising the four space-time components of x'^{μ} and x''^{μ}. Without any loss of generality we might consider $\langle x'|U|x''\rangle$ to be a function of X^{μ} and r^{μ} where

$$X^{\mu} \equiv \tfrac{1}{2}(x'^{\mu} + x''^{\mu}), \qquad r^{\mu} \equiv x'^{\mu} - x''^{\mu}, \tag{4.1}$$

so that

$$\langle x'|U|x''\rangle = \langle X + \tfrac{1}{2}r|U|X - \tfrac{1}{2}r\rangle \equiv U(X, r) \tag{4.2}$$

X^{μ} and r^{μ} will be referred to as Yukawa variables. In the present stage of development X^{μ} is identified to be the center of mass coordinate and r^{μ} and coordinate referring to the internal structure of the field specified by $U(X, r)$.[2] For the present we shall not attempt to give a physical interpretation to these Yukawa variables but shall be simply content to study some mathematical consequences.

14. Local and Non-Local Fields

A field or function U is said, by definition, to be a local field or function if it satisfies the commutation relationship $[x^\mu, U] = 0$; otherwise the field or function is non-local. From this definition it is apparent that if U contains the displacement operator p_μ as well as the space-time operators x^μ it will in general not commute with the x^μ because of (1.1) so that our operator functions are non-local functions. We note that if U_L is a local function

$$\langle x' | U_L | x'' \rangle = U_L(x')\, \delta(x' - x''), \tag{4.3}$$

If we express (4.3) in terms of the Yukawa variables defined by (4.1) we obtain

$$\langle x' | U_L | x'' \rangle = U_L(X)\, \delta(r). \tag{4.4}$$

Consequently,

$$\int \langle x' | U_L | x'' \rangle\, d^4 r = U_L(X). \tag{4.5}$$

(4.5) suggests an averaging process for operator functions which have the property that under the averaging process the average of a local function (function of x^μ) is the C- number obtained by replacing

χ^μ by the Yukawa variable X^μ. In the next section we shall study some properties of this averaging process.

15. The Averaging Process, Moments, Probability Distributions

The average of a non-local function U will be defined to be $\hat{U}(X)$ given by

$$\hat{U}(X) \equiv \int \langle X + r/2 \,|U|\, X - r/2 \rangle \, d^4 r . \tag{4.6}$$

The right hand side of (4,6) reduces to (4,5) for local functions. In general we note that $\hat{U}(X)$ is a function of the so-called center of mass coordinate of the operator function U.

If U can be expanded in terms of the basic functions described in (II-7) we have from (2,37) with $a = 1$

$$X_{n'k'} = (2\pi)^{-2} e^{i n'^\mu (p_\mu - k'_\mu/2)} e^{i k'_\mu x^\mu} , \tag{4.7}$$

for our basic functions, so that

$$\langle x' | X_{n'k'} | x'' \rangle = (2\pi)^{-2} e^{-i n'^\mu k'_\mu/2} e^{i k'_\mu x''^\mu} \delta(x' - x'' + n') , \tag{4.8}$$

whence,

$$\langle X+r/2|X_{n'k'}|X-r/2\rangle = (2\pi)^{-2} e^{-in'^{\mu}k'_{\mu}/2 - ik'_{\mu}r^{\mu}/2 + ik'_{\mu}X^{\mu}} \delta(r+n')$$

$$= (2\pi)^{-2} e^{ik'_{\mu}X^{\mu}} \delta(r+n'), \tag{4.9}$$

but since,

$$U = \sum_{n'k'} Tr(U\widetilde{X}_{n'k'})X_{n'k'}, \tag{4.10}$$

we have from (4.6)

$$\widehat{U}(X) = \int \sum_{n'k'} Tr(U\widetilde{X}_{n'k'})(2\pi)^{-2} e^{ik'_{\mu}X^{\mu}} \delta(r+n') d^4 r$$

$$= \sum_{n'k'} (2\pi)^{-2} Tr(U\widetilde{X}_{n'k'}) e^{ik'_{\mu}X^{\mu}}. \tag{4.11}$$

However,

$$\widetilde{X}_{n'k'} = (2\pi)^{-2} e^{-ik'_{\mu}x^{\mu}} e^{-in'^{\mu}(p_{\mu} - k'_{\mu}/2)}, \tag{4.12}$$

so that

$$(2\pi)^{-2} Tr(U\widetilde{X}_{n'k'}) =$$

$$(2\pi)^{-2} \int \langle p'|U|p''\rangle\langle p''|\widetilde{X}_{n'k'}|p'\rangle d^4p' d^4p'' =$$

$$(2\pi)^{-4} \int \langle p'|U|p'-k'\rangle e^{-in'^{\mu}(p'_{\mu} - k'_{\mu}/2)} d^4p'. \tag{4.13}$$

Consequently, upon introducing (4.13) in (4.11) we obtain

$$\hat{U}(X)=(2\pi)^{-4}\int\sum_{n'k'}\langle p'|U|p'-k'\rangle e^{ik'_\mu x^\mu} e^{-in'^\mu(p'_\mu-k'_\mu/2)}d^4p', \tag{4.14}$$

which from (2.41) and the properties of the δ functions reduces to

$$\hat{U}(X)=\int\langle k'/2|U|-k'/2\rangle e^{ik'_\mu X^\mu} d^4k' \tag{4.15}$$

upon replacing $\sum_{k'}(\ )$ by $\int(\ )d^4k'$. (4.15) presents a convenient expression for computing $\hat{U}(X)$ for special forms.

We will now show that if U is hermitian then $\hat{U}(X)$ is real. Now,

$$\begin{aligned}\hat{\tilde{U}}(X)&=\int\langle k'/2|\tilde{U}|-k'/2\rangle e^{ik'_\mu X^\mu} d^4k'\\ &=\int\langle -k'/2|U|+k'/2\rangle^* e^{ik'_\mu X^\mu} d^4k'\\ &=\left(\int\langle -k'/2|U|k'/2\rangle e^{-ik'_\mu X^\mu} d^4k'\right)^*\\ &=\left(\int\langle k'/2|U|-k'/2\rangle e^{ik'_\mu X^\mu} d^4k'\right)^*\\ &=\left(\hat{U}(X)\right)^*.\end{aligned} \tag{4.16}$$

Hence, if $\tilde{U}=U$, (hermiticity), $\hat{U}(X)$ is real.

The average of U^n will be real if U is hermitian and is obtained by replacing U in (4.15) by U^n to obtain

$$\widehat{U^n}(X) = \int \langle k'/2 | U^n | -k'/2 \rangle e^{ik'_\mu X^\mu} d^4 k'. \qquad (4.17)$$

We shall define $\widehat{U^n}(X)$ to be the n^{th} moment relative to the averaging process defined in this section. Now from the theory of statistics we can state that if the moments relative to a certain averaging process are known then the probability distribution can be ascertained in the following manner. Let us define

$$M(t, U) \equiv \widehat{e^{itU}}, \qquad (4.18)$$

to be the characteristic function of U relative to our averaging process. The probability distribution of $\underline{U}$ is

$$P(\underline{U}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-it\underline{U}} M(t, U)\, dt. \qquad (4.19)$$

In order that (4.19) be satisfactory we must show that the moments implied by (4.19) are the same as those given by (4.17):

$$\int_{-\infty}^{+\infty} \underline{U}^n P(\underline{U})\, d\underline{U} = \widehat{U^n}(X). \qquad (4.20)$$

Upon putting (4.18) in (4.20) and using (4.15) with U replaced by e^{itU} we obtain

$$\int_{-\infty}^{+\infty} \underline{U}^n P(\underline{U})\, d\underline{U} =$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \underline{U}^n \int_{-\infty}^{+\infty} e^{-it\underline{U}} \int \langle k'/2 | e^{itU} | -k'/2 \rangle e^{ik'_\mu x^\mu} d^4k'\, dt\, d\underline{U}. \tag{4.21}$$

But,

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \underline{U}^n e^{-it\underline{U}} d\underline{U} = \frac{(i)^n}{2\pi} \frac{\partial^n}{\partial t^n} \int_{-\infty}^{+\infty} e^{-it\underline{U}} d\underline{U}$$

$$= (i)^n \frac{\partial^n}{\partial t^n} \delta(t), \tag{4.22}$$

where $\delta(t)$ is a one dimensional Dirac function. Consequently, (4.21) becomes

$$\int_{-\infty}^{+\infty} \underline{U}^n P(\underline{U})\, d\underline{U} =$$

$$\int_{-\infty}^{+\infty} \int \langle k'/2 | e^{itU} | -k'/2 \rangle e^{ik'_\mu x^\mu} (i)^n \frac{\partial^n}{\partial t^n} \delta(t)\, d^4k'\, dt, \tag{4.23}$$

Upon integrating (4.23) by parts n times with respect to t and making use of the integral property of the $\delta-$ function we obtain

$$\int_{-\infty}^{+\infty} \underset{\sim}{U}^n P(\underset{\sim}{U}) d\underset{\sim}{U} = \int \langle \tfrac{k'}{2} | U^n | -\tfrac{k'}{2} \rangle e^{ik'_\mu X^\mu} d^4 k'$$

$$= \widehat{U^n}(X), \tag{4.24}$$

from (4.17). Thus $P(\underset{\sim}{U})$ possesses the property envisioned, namely, that stated in (4.20). It must be emphasized that knowledge of $P(\underset{\sim}{U})$ is not equivalent to the knowledge of the operator U.

Another equivalent form for $\hat{U}(X)$ may be obtained by introducing the reflection operator $S_\alpha^2 \equiv R$, (3.61) with matrix elements

$$\langle p' | R | p'' \rangle = \delta(p' + p''). \tag{4.25}$$

We will show that

$$\hat{U}(X) = 16 \,\mathrm{Tr}\, U R e^{2iX^\mu p_\mu}. \tag{4.26}$$

Now,

$$16 \,\mathrm{Tr}\, U R e^{2iX^\mu p_\mu} = 16 \iiint \langle p' | U | p''' \rangle \langle p''' | R | p'' \rangle \langle p'' | e^{2iX^\mu p_\mu} | p' \rangle \times$$

$$d^4 p''' d^4 p'' d^4 p'$$

$$= 16 \iint \langle p' | U | -p'' \rangle e^{2iX^\mu p'_\mu} \delta(p'' - p') d^4 p'' d^4 p'$$

$$= 16 \int \langle p' | U | -p' \rangle e^{2iX^\mu p'_\mu} d^4 p'$$

$$= \int \langle \tfrac{k'}{2} | U | -\tfrac{k'}{2} \rangle e^{ik'_\mu X^\mu} d^4 k'.$$

The above consequently verifies (4.26) so that (4.15) and (4.6) may be written as (4.26). Our ability to write $\widehat{U}(X)$ as a trace indicates that $\widehat{U}(X)$ is independent of the representation of the operators x and p. Moreover, we may state that $\widehat{SUS^{-1}} = \widehat{U}$ if S commutes with $Re^{2ix^\mu p_\mu}$ in (4.26).

16. $\widehat{U}(X)$ for Special Forms

In this section we will calculate $\widehat{U}(X)$ for various forms making use of (4.15). If we replace U in (4.15) by $f(p)e^{ik_\mu x^\mu}$ we may readily show that

$$\widehat{f(p)e^{ik_\mu x^\mu}} = f(k/2)e^{ik_\mu X^\mu}, \tag{4.27}$$

and in a similar fashion we can show that

$$\widehat{e^{ik_\mu x^\mu} f(p)} = f(-k/2)e^{ik_\mu X^\mu}. \tag{4.28}$$

(4.27) and (4.28) enables one to compute expressions of type $\widehat{f(p)x^{\alpha_1}x^{\alpha_2}\ldots x^{\alpha_n}}$ and $\widehat{x^{\alpha_1}x^{\alpha_2}\ldots x^{\alpha_n}f(p)}$ by simply differentiating both sides of (4.27) and (4.28) partially with respect to $k_{\alpha_1}, k_{\alpha_2}, \ldots k_{\alpha_n}$ and then setting the k's equal to zero. For example

$$\widehat{f(p)x^\mu} = \tfrac{1}{2i}\,\frac{\partial f(k)}{\partial k_\mu}\Big|_{k=0} + X^\mu f(0),$$
$$\widehat{x^\mu f(p)} = -\tfrac{1}{2i}\,\frac{\partial f(k)}{\partial k_\mu}\Big|_{k=0} + X^\mu f(0). \tag{4.29}$$

To compute $\widehat{e^{ic^\mu p_\mu} g(x)}$ it is easier to appeal to (4.6). Straightforward calculation yields

$$\widehat{e^{ic^\mu p_\mu} g(x)} = g(X + c/2). \tag{4.30}$$

In a similar fashion

$$\widehat{g(x)e^{ic^\mu p_\mu}} = g(X - c/2). \tag{4.31}$$

(4.30) and (4.31) are useful for calculating expressions of type $\widehat{p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_n} g(x)}$ or $\widehat{g(x) p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_n}}$ respectively. This is effected by differentiating both sides of (4.30) and (4.31) with respect to $c^{\alpha_1}, c^{\alpha_2}, \cdots c^{\alpha_n}$ and then setting the c's equal to zero.

$$\widehat{p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_n} g(x)} = \left(\frac{1}{2i}\right)^n \frac{\partial^n}{\partial X^{\alpha_1}\partial X^{\alpha_2}\cdots\partial X^{\alpha_n}}\, g(X),$$
$$\widehat{g(x) p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_n} g(x)} = \left(-\frac{1}{2i}\right)^n \frac{\partial^n}{\partial X^{\alpha_1}\partial X^{\alpha_2}\cdots\partial X^{\alpha_n}}\, g(X), \tag{4.32}$$

follows from the above prescription.

Let us now consider $\widehat{e^{ic\cdot p}U}$. From (4.26) we may write

$$\widehat{e^{ic\cdot p}U} = 16\,\mathrm{Tr}\, e^{ic^\mu p_\mu} U R e^{2iX^\mu p_\mu}, \tag{4.33}$$

which becomes upon using the cyclic property of the trace

$$\widehat{e^{ic\cdot p}U} = 16\,\mathrm{Tr}\, U R e^{2i(X^\mu + c^\mu/2)p_\mu}. \tag{4.34}$$

Consequently,

$$\widehat{e^{ic\cdot p}U} = \hat{U}(X + c/2). \tag{4.35}$$

(4.35) has the same basic structure as (4.30) so that proceeding in the same manner as we did to obtain (4.32) we obtain

$$\begin{aligned} \widehat{p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_n} U} &= \left(\frac{1}{2i}\right)^n \frac{\partial^n}{\partial X^{\alpha_1} \partial X^{\alpha_2} \cdots \partial X^{\alpha_n}} \hat{U}(X), \\ \widehat{U p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_n}} &= \left(-\frac{1}{2i}\right)^n \frac{\partial^n}{\partial X^{\alpha_1} \partial X^{\alpha_2} \cdots \partial X^{\alpha_n}} \hat{U}(X). \end{aligned} \tag{4.36}$$

The second equation of (4.36) follows in a similar fashion by noting that

$$\widehat{Ue^{ic^\mu p_\mu}} = 16 \,\mathrm{Tr}\, Ue^{ic^\mu p_\mu} Re^{2iX^\mu p_\mu}$$

$$= 16 \,\mathrm{Tr}\, URe^{-ic^\mu p_\mu} e^{2iX^\mu p_\mu}$$

$$= \hat{U}(X - c/2).$$

General expressions for $\widehat{x^{\alpha_1} x^{\alpha_2} \ldots x^{\alpha_n} U}$ and $\widehat{U x^{\alpha_1} x^{\alpha_2} \ldots x^{\alpha_n}}$ of the type (4.36) when the p's are replaced by the x's do not seem to exist. However, we may obtain some useful expressions in the following manner. From (4.26) we may write

$$\widehat{e^{ik_\mu x^\mu} U} = 16 \,\mathrm{Tr}\, e^{ik_\mu x^\mu} URe^{2iX^\mu p_\mu}$$

$$= 16 \,\mathrm{Tr}\, URe^{2iX^\mu p_\mu} e^{ik_\mu x^\mu}. \qquad (4.37)$$

We also have

$$\widehat{Ue^{ik_\mu x^\mu}} = 16 \,\mathrm{Tr}\, Ue^{ik_\mu x^\mu} Re^{2iX^\mu p_\mu}$$

$$= 16 \,\mathrm{Tr}\, URe^{-ik_\mu x^\mu} e^{2iX^\mu p_\mu}$$

$$= 16 \,\mathrm{Tr}\, URe^{2iX^\mu (p_\mu + k_\mu)} e^{-ik_\mu x^\mu}$$

$$= 16 \,\mathrm{Tr}\, URe^{2iX^\mu p_\mu} e^{ik_\mu (2X^\mu - x^\mu)}$$

$$= \widehat{e^{ik_\mu (2X^\mu - x^\mu)} U}, \qquad (4.38)$$

if we recall $Rf(x) = f(-x)R$, $e^{ik_\mu x^\mu} f(x,p) e^{-ik_\mu x^\mu} = f(x, p-k)$ (2.39), the properties of the trace and comparison with (4.37). (4.38) establishes the identity

$$\widehat{U x^\mu + x^\mu U} = 2 X^\mu \hat{U}, \tag{4.39}$$

as we see upon differentiating both sides of (4.38) partially with respect to k_μ and then setting the k's equal to zero. Further partial differentiations with respect to k_μ and subsequent evaluation with the k's zero lead to identities of type similar to (4.39). If in (4.39) U is replaced by $U x^\nu + x^\nu U \equiv [x^\nu, U]_+$, we obtain

$$\widehat{[x^\mu, [x^\nu, U]_+]_+} = 4 X^\mu X^\nu \hat{U}. \tag{4.40}$$

Now if in (4.38) we replace U by $e^{-ik_\mu x^\mu} U$ and then replace the k's by their negatives we obtain

$$\widehat{e^{ik_\mu x^\mu} U e^{-ik_\mu x^\mu}} = \widehat{e^{2ik_\mu (x^\mu - X^\mu)} U}. \tag{4.41}$$

Upon differentiating both sides of (4.41) partially with respect to k_μ and then setting k_μ equal to zero we obtain

$$\widehat{x^\mu U} = X^\mu \hat{U} + \tfrac{1}{2} \widehat{[x^\mu, U]_-}. \tag{4.42}$$

Upon using (4.42) in conjunction with (4.39) we may also write

$$\widehat{U\chi^\mu} = X^\mu \hat{U} - \tfrac{1}{2}\,\widehat{[\chi^\mu, U]_-}\,. \tag{4.43}$$

Proceeding further in this manner we may establish relationships between $\widehat{\chi^{\alpha_1}\chi^{\alpha_2}\ldots\chi^{\alpha_n}U}$ and the commutators of the χ's with the U and similarly for $\widehat{U\chi^{\alpha_1}\chi^{\alpha_2}\ldots\chi^{\alpha_n}}$.

Before going on to the next section let us examine a consequence of the second line of (4.38) with $X^\mu \longrightarrow -\tfrac{1}{2}X^\mu$ on the left side

$$\widehat{Ue^{ik_\mu \chi^\mu}}(-X/2) =$$

$$16\,\mathrm{Tr}\,URe^{-ik_\mu \chi^\mu}e^{-iX^\mu p_\mu}, \tag{4.44}$$

It will be noticed that (4.44) contains aside from a normalization factor the adjoint basic functions defined by (2.37) if we replace x'^μ by X^μ and set $a = 0$. Consequently, (4.44) may be written as

$$\widehat{Ue^{ik_\mu \chi^\mu}}(-X/2) = 2^4(2\pi)^2\,\mathrm{Tr}\,UR\bar{X}_{xk}\,. \tag{4.45}$$

But $\mathrm{Tr}\, UR\bar{X}_{xk}$ is the representative of the second kind of relative to the set of basic functions $\bar{X}_{xk}$ so that from (2.2) with the U's replaced by the X's,

$$UR = \frac{1}{2^4(2\pi)^2}\iint \widehat{Ue^{ik_\mu x^\mu}}(-X/2)\, X_{xk}\, d^4k\, d^4X, \qquad (4.46)$$

which becomes upon noting that $R^2 = 1$,

$$U = \frac{1}{2^4(2\pi)^2}\iint \widehat{Ue^{ik_\mu x^\mu}}(-X/2) X_{xk}\, R\, d^4k\, d^4X. \qquad (4.47)$$

(4.47) implies that knowledge of $\widehat{Ue^{ik_\mu x^\mu}(X)}$ implies knowledge of the operator U.

17. The Probability Distribution of Operators Relative to the Averaging Process

It is of some interest to carry out the averaging process of (IV-15) for operators whose properties are well known. In particular it will be instructive to consider those operators for which it would be possible to obtain closed expressions for their probability distributions under the averaging process. Amongst such operators are included any local function of the space-time operators and any non-local function of the displacement operators only. In the former case we must conclude from (4.30) with $C = 0$, that the moments of $[g(x)]^n$ are equal to $[g(x)]^n$ which implies from

(4.18) that $M(t, g(x))$ is equal to $\exp itg(X)$ which tells us according to (4.19) that

$$P(\underline{g(x)}) = \delta(\underline{g(x)} - g(X)). \tag{4.48}$$

In the latter case for non-local functions of the displacement operators alone, say $f(p)$, we conclude from (4.27) with $k = 0$, and (4.18) in conjunction with (4.19) that

$$P(\underline{f(p)}) = \delta(\underline{f(p)} - f(0)). \tag{4.49}$$

(4.48) implies that measurements of a local function lead to one and only one value--*if* indeed we might make an association of our averaging process with anything corresponding to physical reality in this stage of the development. The implications of (4.49) are also similar but we observe that for this case the distribution of $\underline{f(p)}$ centers about $f(0)$ which is devoid of reference to the displacement operator. This is to be contrasted with the results of (4.48) where the distribution of $\underline{g(x)}$ centers about $g(X)$ which depends upon our external Yukawa variable X^μ.

These results which are direct consequences of our averaging process for the extreme cases of a local function and a non-local function which depends solely upon the displacement operators are to

a certain extent plausible. Let us now consider what happens when we treat the non-local function L_3 defined by

$$L_3 \equiv (x_1 p_2 - x_2 p_1). \tag{4.50}$$

Now,

$$\begin{aligned}\widehat{L_3 U} &= \widehat{(x_1 p_2 - x_2 p_1) U} \\ &= \frac{-i}{2}\left(X_1 \frac{\partial \hat{U}}{\partial X_2} - X_2 \frac{\partial \hat{U}}{\partial X_1}\right) + \frac{1}{2}\widehat{p_2 [x_1, U]} \\ &\qquad - \frac{1}{2}\widehat{p_1 [x_2, U]},\end{aligned} \tag{4.51}$$

as we see upon applying (4.36) and (4.42). If we apply (4.36) to the last two terms of (4.51) we finally obtain

$$\widehat{L_3 U} = \frac{-i}{2}\left(X_1 \frac{\partial \hat{U}}{\partial X_2} - X_2 \frac{\partial \hat{U}}{\partial X_1}\right) - \frac{i}{4}\left\{\frac{\partial}{\partial X_2}\widehat{[x_1, U]} - \frac{\partial}{\partial X_1}\widehat{[x_2, U]}\right\}. \tag{4.52}$$

Consequently,

$$\widehat{L_3^n} = \frac{-i}{2}\left(X_1 \frac{\partial \widehat{L_3^{n-1}}}{\partial X_2} - X_2 \frac{\partial \widehat{L_3^{n-1}}}{\partial X_1}\right) - \frac{i}{4}\left\{\frac{\partial}{\partial X_2}\widehat{[x_1, L_3^{n-1}]} - \frac{\partial}{\partial X_1}\widehat{[x_2, L_3^{n-1}]}\right\} \tag{4.53}$$

Now from (4.52) we see that if $U = 1$, $\hat{L}_3 = 0$. Moreover,

if we put $U = L_3$ therein and make use of the well known relations

$$[\chi_1, L_3] = -i\chi_2 \qquad [\chi_2, L_3] = i\chi_1 \tag{4.54}$$

and recall that $\widehat{\chi}_1$ and $\widehat{\chi}_2$ are respectively equal to X_1 and X_2 from (4.5), we may verify that $\widehat{L_3^2} = -\frac{1}{2}$ for we then have

$$\widehat{L_3^2} = -\frac{1}{4}\left\{\frac{\partial}{\partial X_2} X_2 + \frac{\partial}{\partial X_1} X_1\right\} = -\frac{1}{2} \tag{4.55}$$

The fact that $\widehat{L_3^2}$ is non-positive indicates that the averaging process as developed above is not a good one. We shall, however, investigate the averaging process further.

It is well known that the spectra of an operator is invariant under a unitary transformation. The question arises as to what changes are brought about on the probability distribution of an operator relative to our averaging process if it is subject to a unitary transformation. A further inquiry might be made as to the condition that the probability distribution be unaltered. Let us give our attention to any operator U. Now if S is a unitary operator $S = S^{-1}$, then from (4.26)

$$\widehat{SUS^{-1}} = 16\,\mathrm{Tr}\, SUS^{-1} R e^{2ix^\mu p_\mu}, \tag{4.56}$$

so that it is at once apparent that the probability distribution of $\underline{U}$ is indeed subject to change and that it will be invariant if S commutes with U or with $Re^{2ix^\mu p_\mu}$. If we assume that the unitary operator S may be written as

$$S = e^{ig}, \tag{4.57}$$

where g is hermitian, we may state that the probability distribution of $\underline{U}$ is invariant if g commutes with U or with $Re^{2ix^\mu p_\mu}$. Let us investigate the latter possibility.

$$[g, Re^{2ix^\mu p_\mu}] = 0, \qquad \text{or}$$

$$gRe^{2ix^\mu p_\mu} - Re^{2ix^\mu p_\mu} g = 0, \tag{4.58}$$

implies upon taking the adjoint and recalling that $\bar{g} = g$, $\bar{R} = R$,

$$e^{-2ix^\mu p_\mu} Rg - ge^{-2ix^\mu p_\mu} R = 0, \tag{4.59}$$

(4.59) is consistent with (4.58) since $Re^{2ix^\mu p_\mu} = +e^{-2ix^\mu p_\mu} R$. Now since $R^2 = 1$, (4.58) or (4.59) implies

$$g = e^{-2ix^\mu p_\mu} RgRe^{2ix^\mu p_\mu} \tag{4.60}$$

This latter equation tells us that

$$g = R e^{2ix^\mu p_\mu} g e^{-2ix^\mu p_\mu} R, \tag{4.61}$$

since p_μ anticommutes with R. (4.61) yields equivalently

$$R g R = e^{2ix^\mu p_\mu} g e^{-2ix^\mu p_\mu} \tag{4.62}$$

which tells us that the probability distribution of $\underline{U}$ will be invariant if

$$R g(x,p) R = g(x+X,p), \quad \text{or}$$

$$g(-x,-p) = g(x+X,p), \tag{4.63}$$

from the property of the reflection operator R and (3.34). (4.63) can be satisfied if g is an even function of the p's only. This result is quite general so that we can state that for any operator $U, \hat{U}$ and consequently the probability distribution of $\underline{U}$ will be the same for the operator $\exp[ig_e(p)U]\exp[-ig_e(p)]$, where $g_e(p)$ is an even function of the p's.

We shall now undertake to investigate the second possibility for the invariance of the probability distribution of any operator

For this case we must have

$$[g, U] = 0, \tag{4.64}$$

so that g will depend upon the operator U under consideration. To effect a solution of (4.64) we could appeal to the work of Part III section 11. However, we may effect a solution by inspection by noting that the hermitian expression,

$$g = \sum_{U'} a(U') |U'\rangle \langle U'|, \tag{4.65}$$

where the a's are real and $|U'\rangle, \langle U'|$ are the eigen-kets and eigen-bras corresponding to our operator U, satisfies (4.64). The operators $|U'\rangle\langle U'|$ form a mutually commuting system. (4.65) may be written as

$$g = \sum_{U'} a(U') g_{U'}, \tag{4.66}$$

where

$$g_{U'} \equiv |U'\rangle \langle U'|. \tag{4.67}$$

We note that each of the g's satisfies

$$g_{U'}^{n} = g_{U'} ,$$
$$g_{U'} g_{U''} = g_{U'} \delta_{U'U''} . \tag{4.68}$$

where n is any integer. This is due to the property: $\langle U' | U'' \rangle = \delta_{U'U''}$ which is satisfied by our eigen-bras and kets. (4.68) enables us to write

$$S_{U'} \equiv e^{i a(U') g_{U'}}$$
$$= 1 + (e^{i a(U')} - 1) g_{U'} . \tag{4.69}$$

Consequently,

$$S_{U} \equiv e^{i \sum_{U'} a(U') g_{U'}}$$
$$= 1 + \sum_{U'} (e^{i a(U')} - 1) g_{U'} , \tag{4.70}$$

as we see upon using (4.68) again. Now if A is any operator we may readily show that

$$S_{U'} A S_{U'}^{-1} = A + \operatorname{Sin.} a(U') [g_{U'}, A] - 2 \operatorname{Sin.}^2 \tfrac{1}{2} a(U') [g_{U'}, g_{U'}, A] . \tag{4.71}$$

In general

$$S_U A S_U^{-1} = A + i\sum_{U'} \mathrm{Sin.}\,a(U')[g_{U'}, A]$$

$$-\frac{1}{2}\sum_{U',U''}[1-\mathrm{Cos}\,a(U')-\mathrm{Cos}\,a(U'')+\mathrm{Cos}(a(U')-a(U''))][g_{U'},[g_{U''},A]]$$

$$+i\sum_{U',U''}[\mathrm{Sin.}(a(U')-a(U''))-\mathrm{Sin.}\,a(U')+\mathrm{Sin.}\,a(U'')]g_{U'}Ag_{U''}. \quad (4.72)$$

(4,72) would enable us to compute the change of $\hat{A}$ brought about by a unitary transformation which renders $\hat{U}$ and consequently the probability distribution of $\underset{\sim}{U}$ invariant. The first type of transformation exemplified by the unitary operator $S_\ell = \exp i g_\ell(p)$ leaves all $\hat{A}$ invariant so that only the second case with our unitary operator given by (4.70) is of importance in ascertaining changes in the probability distribution of operators induced by unitary transformations which render $\hat{U}$ invariant.

18. Examination of Results

The results that we have obtained particularly that one which pertains to the non-positive nature of $\widehat{L_3^2}$ where L_3 is given by (4.50) does not appear to be reasonable. This leads us to venture the statement that the averaging process is perhaps incorrect. From (4.26) the density operator function is

$$\rho \equiv 16\, R e^{2ix^\mu p_\mu} \tag{4.73}$$

and our results are intimately related to this identification.

Now there exists a striking analogy between our averaging process and the averaging process which is used in Quantum Statistics[1] where

$$\rho_Q \equiv \sum_{m'} | m' \rangle P_{m'} \langle m' | \tag{4.74}$$

and

$$\sum_{m'} P'_m = 1 . \tag{4.75}$$

If we introduce (4.74) in (4.18) and (4.19) with the operator $\widehat{U}$ replaced by $Tr\, \rho_Q\, U$ we obtain upon using (4.70)

$$P(\rho_Q) = \sum_{m'} P_{m'}\, \delta(\underline{\rho_Q} - P_{m'}), \tag{4.76}$$

a very reasonable result, which may be interpreted to mean that measurements of ρ_Q lead to any one of the eigen-values of ρ_Q, namely, the $P_{m'}$. The introduction of (4.73) into (4.18) and (4.19), however, leads to divergent results since the trace of any even power of ρ is infinite. It is believed that our averaging process

should possess the same kind of consistency as that exhibited by the density operator ρ_Q defined in (4.74) so that if we accept this test for consistency we must conclude that ρ given by (4.74) is defective.

Indeed, it may not be amiss to redefine our averaging process and use (4.74) in place of (4.73) with our $|m'\rangle$ denoting an arbitrary complete set of basic kets. It does not seem that one would be able to choose such a ρ which would have properties similar to that exhibited by (4.5). If such were the case our formalism would make no reference to the Yukawa variables (4.1) and subsequent averaging over the internal variable r_μ. The averaging process seems to be inconsistent with the notion of eigenvalues of operators so that this may be cited as an objection to the averaging formalism in the present state of development of the theory. It may be necessary to resort to the ρ_Q given by (4.74) with $P_{m'}$ denoting the probability distribution of the m' which could stand for the collection of Quantum numbers denoting the momenta and the mass. The kets $|m'\rangle$ can be associated with the basic kets of a four dimensional harmonic oscillator, for example. It is planned to pursue this line of thought further.

V. VARIATION PRINCIPLE FOR OPERATOR FIELDS

19. First Operator Identity

Let us consider the trace G of an operator function $L(f_A)$ of a set of operators f_A with $A = 1, 2, \ldots$

$$G = \mathrm{Tr}\, L(f_A). \tag{5.1}$$

If each of the f_A's is subject to an arbitrary variation δf_A (operators in general) the first variation of G may be written as

$$\delta G = \mathrm{Tr}\, F^A(f_A)\, \delta f_A , \tag{5.2}$$

where the F^A may be calculated if the explicit form of $L(f_A)$ is specified. Moreover, in (5.2) we are using the summation convention for the indices A. Now if the variation δf_A arises from an arbitrary infinitesimal similarity transformation: $U \rightarrow e^g U e^{-g}$,

$$\delta_s f_A \simeq [g, f_A], \tag{5.3}$$

if g is an arbitrary infinitesimal operator. In (5.3) we have denoted the change of f_A under the infinitesimal similarity transformation by $\delta_s f_A$. Upon replacing δf_A by $\delta_s f_A$ given by (5.3) in (5.2) we obtain to first order

$$\delta_S G = \mathrm{Tr}\, F^A [g, f_A] = \mathrm{Tr}\,[f_A, F^A]\, g . \tag{5.4}$$

But G is invariant under a similarity transformation. $\delta_S G$ in (5.4) denotes the "change" of G under the similarity transformation. Consequently, since $\delta_S G = 0$, and since g is an arbitrary infinitesimal operator we must conclude that

$$[f_A F^A] \equiv 0 . \tag{5.5}$$

The above equation establishes an identity (the first identity) between the coefficients F^A appearing in (5.2) and the f_A. (5.5) would enable one to construct identities corresponding to any function of the operator f_A, namely, $L(f_A)$. This result is quite general and we will have occasion to use it in later developments.

20. Integration of Operators

In order to make connection with certain mathematical operations in current physical theory it is convenient to introduce an operation which enables us to integrate operators. Such a type of integration may be brought about by introducing a four-vector space-time C- number C^μ into our definition in the following manner

$$\int e^{ic^\mu p_\mu} A(p,x)\, e^{-ic^\mu p_\mu}\, dC, \tag{5.6}$$

where $A(p, x)$ represents an operator and dC corresponds to any combination of the products of dc^0, dc^1, dc^2 and dc^3.

For example let us consider the operator N defined by

$$N[\sigma] \equiv \int_\sigma e^{ic^\mu p_\mu} N^\nu e^{-ic^\mu p_\mu} d\sigma_\nu(c), \tag{5.7}$$

where σ is a space-like surface in the $C-$ space. The operator $N[\sigma]$ given by (5.7) is in general a function of the surface σ so that

$$\begin{aligned} \frac{\delta N[\sigma]}{\delta\sigma(c)} &= \frac{\partial}{\partial c^\nu} e^{ic^\mu p_\mu} N^\nu e^{-ic^\mu p_\mu}, \\ &= i\, e^{ic^\mu p_\mu} [p_\nu, N^\nu] e^{-ic^\mu p_\mu}, \end{aligned} \tag{5.8}$$

where the left hand side of (5.8) is the functional derivative of $N[\sigma]$. If $[p_\nu, N^\nu] = 0$, then $N[\sigma]$ is independent of σ. Thus, $[p_\nu, N^\nu] = 0$ provides us with a useful criterion for the independence of surface integrals of type (5.7) upon the shape of our surface. Furthermore, this example indicates that our definition of integration enables us to take over most of the properties of integrals occurring in the $C-$ number formalism.

21. Infinitesimal C- Number Transformations

A covariant vector operator, say A_μ, is an operator which transforms like

$$A_\mu(c) = \left(\partial_0 C^\alpha / \partial C^\mu\right)_0 A_\alpha \tag{5.9}$$

under the C- number transformation

$$C^\mu \equiv C^\mu({}_0C^\alpha). \tag{5.10}$$

(5.9) establishes the relationship existing between the "components" of our covariant vector for different C- number coordinate systems. In general a mixed tensor operator, say $T^{\alpha_1,\alpha_2\cdots}_{\beta_1,\beta_2\cdots}$, transforms like

$$T^{\alpha_1,\alpha_2\cdots}_{\beta_1,\beta_2\cdots}(c) = \left(\partial C^{\alpha_1}/\partial_0 C^{\mu_1}\, \partial C^{\alpha_2}/\partial_0 C^{\mu_2}\cdots\right) \times$$

$$\left(\partial_0 C^{\nu_1}/\partial C^{\beta_1}\, \partial_0 C^{\nu_2}/\partial C^{\beta_2}\cdots\right)_0 T^{\mu_1,\mu_2\cdots}_{\nu_1,\nu_2\cdots}, \tag{5.11}$$

(5.11) is merely the conventional definition of tensor transformations.

If we consider the transformation (5.10) to be an infinitesimal C- number transformation

$$C^\mu \equiv {}_0C^\mu + \xi^\mu({}_0C^\alpha), \tag{5.12}$$

where ξ^{μ} is an arbitrary $C-$ number function of the ${}_{\circ}C^{\alpha}$, then the first order equations corresponding to (5.11) become

$$T^{\alpha_1 \alpha_2 \cdots}_{\beta_1 \beta_2 \cdots}(C) = {}_{\circ}T^{\alpha_1 \alpha_2 \cdots}_{\beta_1 \beta_2 \cdots} + \xi^{\alpha_1}_{,\mu_1}\, {}_{\circ}T^{\mu_1 \alpha_2 \cdots}_{\beta_1 \beta_2 \cdots} + \xi^{\alpha_2}_{,\mu_2}\, {}_{\circ}T^{\alpha_1 \mu_2 \cdots}_{\beta_1 \beta_2 \cdots}$$

$$- \xi^{\nu_1}_{,\beta_1}\, {}_{\circ}T^{\alpha_1 \alpha_2 \cdots}_{\nu_1 \beta_2 \cdots} - \xi^{\nu_2}_{,\beta_1}\, {}_{\circ}T^{\alpha_1 \alpha_2 \cdots}_{\beta_1 \nu_2 \cdots} \cdots \qquad (5.13)$$

However,

$$T^{\alpha_1 \alpha_2 \cdots}_{\beta_1 \beta_2 \cdots}(C) = T^{\alpha_1 \alpha_2 \cdots}_{\beta_1 \beta_2 \cdots}({}_{\circ}C) + \xi^{\mu}\, {}_{\circ}T^{\alpha_1 \alpha_2 \cdots}_{\beta_1 \beta_2 \cdots , \mu} \qquad (5.14)$$

to first order, where in (5.13) and (5.14) we are using $(\ )_{,\mu}$ to denote partial differentiation with respect to ${}_{\circ}C^{\mu}$. The first variation of the tensor $T^{\alpha_1 \alpha_2 \cdots}_{\beta_1 \beta_2 \cdots}$ which arises from an infinitesimal deformation of our $C-$ number coordinate mesh system is defined to be

$$\delta_m T^{\alpha_1 \alpha_2 \cdots}_{\beta_1 \beta_2 \cdots} \equiv T^{\alpha_1 \alpha_2 \cdots}_{\beta_1 \beta_2 \cdots}({}_{\circ}C) - {}_{\circ}T^{\alpha_1 \alpha_2 \cdots}_{\beta_1 \beta_2 \cdots}$$

$$= {}_{\circ}T^{\mu \alpha_2 \cdots}_{\beta_1 \beta_2 \cdots}\, \xi^{\alpha_1}_{,\mu} + {}_{\circ}T^{\alpha_1 \mu \cdots}_{\beta_1 \beta_2 \cdots}\, \xi^{\alpha_2}_{,\mu} \cdots$$

$$- {}_{\circ}T^{\alpha_1 \alpha_2 \cdots}_{\nu \beta_1 \cdots}\, \xi^{\nu}_{,\beta_1} - {}_{\circ}T^{\alpha_1 \alpha_2 \cdots}_{\beta_1 \nu \cdots}\, \xi^{\nu}_{,\beta_2} \cdots \qquad (5.15)$$

$$- {}_{\circ}T^{\alpha_1 \alpha_2 \cdots}_{\beta_1 \beta_2 \cdots , \mu}\, \xi^{\mu},$$

as we see upon introducing (5.14) in (5.13) and redefining our dummy indices. It is not difficult to show that $\delta_m T^{\alpha_1 \alpha_2 \cdots}_{\beta_1 \beta_2 \cdots}$ is a tensor if ξ^μ is a vector. Indeed (5.15) may be written as

$$\delta_m T^{\alpha_1 \alpha_2 \cdots}_{\beta_1 \beta_2 \cdots} = {}_0T^{\mu \alpha_2 \cdots}_{\beta_1 \beta_2 \cdots}\, \xi^{\alpha_1}_{;\mu} + {}_0T^{\alpha_1 \mu \cdots}_{\beta_1 \beta_2 \cdots}\, \xi^{\alpha_2}_{;\mu} \cdots$$

$$- {}_0T^{\alpha_1 \alpha_2 \cdots}_{\nu \beta_2 \cdots}\, \xi^{\nu}_{;\beta_1} - {}_0T^{\alpha_1 \alpha_2}_{\beta_1 \nu}\, \xi^{\nu}_{;\beta_2} \cdots$$

$$- {}_0T^{\alpha_1 \alpha_2 \cdots}_{\beta_1 \beta_2 \cdots ;\mu}\, \xi^\mu , \qquad (5.16)$$

where $(\ \ )_{;\mu}$ denotes covariant differentiation defined with respect to a set, ${}_0\Gamma^{\mu}_{\nu\sigma}$, of Christofel symbols. We observe that this set is quite arbitrary and may be calculated using any metric tensor. Generally the set ${}_0\Gamma^{\mu}_{\nu\sigma}$ can correspond to any linear connection.

Now if $\underset{\sim}{T}^{\alpha_1 \alpha_2 \cdots}_{\beta_1 \beta_2 \cdots}$ is a mixed tensor of weight one (tensor density) the transformation equations will be similar to (5.11) with the exception of a factor on the right hand side which is the Jacobian of the transformation (5.10). For an infinitesimal transformation (5.12) the Jacobian $J(C/{}_0C)$ or $J({}_0C/C)$ works out to be up to the first order

$$J(C/{}_0C) = 1 + \xi^\mu_{,\mu} ,$$
$$J({}_0C/C) = 1 - \xi^\mu_{,\mu} . \qquad (5.17)$$

Proceeding in a fashion similar to that which lead to (5.15) after noting that the transformation (6.11) is multiplied by $J\left({}^{o}\!/_{c}\right)$ on the right hand side we obtain

$$\delta_m \underset{\sim}{T}{}^{\alpha_1\alpha_2\cdots}_{\beta_1\beta_2\cdots} = {}_o\underset{\sim}{T}{}^{\mu\,\alpha_2\cdots}_{\beta_1\beta_2\cdots}\,\xi^{\alpha_1}_{,\mu} + {}_o\underset{\sim}{T}{}^{\alpha_1\mu\cdots}_{\beta_1\beta_2\cdots}\,\xi^{\alpha_2}_{,\mu}\cdots$$
$$- {}_o\underset{\sim}{T}{}^{\alpha_1\alpha_2\cdots}_{\nu\,\beta_2\cdots}\,\xi^{\nu}_{,\beta_1} - {}_o\underset{\sim}{T}{}^{\alpha_1\alpha_2\cdots}_{\beta_1\nu\cdots}\,\xi^{\nu}_{,\beta_2}\cdots \tag{5.18}$$
$$- {}_o\underset{\sim}{T}{}^{\alpha_1\alpha_2\cdots}_{\beta_1\beta_2\cdots,\mu}\,\xi^{\mu} - {}_o\underset{\sim}{T}{}^{\alpha_1\alpha_2\cdots}_{\beta_1\beta_2\cdots}\,\xi^{\mu}_{,\mu}\,,$$

for the change of our tensor density of weight one corresponding to an infinitesimal deformation of our $C-$ number mesh system.

Examination of (5.15) and (5.18) shows that we may write

$$\delta_m T_A = -\,\omega^{B\mu}_{A\nu}\,\xi^{\nu}_{,\mu}\;{}_oT_B - {}_oT_{A,\mu}\,\xi^{\mu}, \tag{5.19}$$

and

$$\delta_m \underset{\sim}{T}_A = -\,\omega^{B\mu}_{A\nu}\,\xi^{\nu}_{,\mu}\;{}_o\underset{\sim}{T}_B - {}_o\underset{\sim}{T}_{A,\mu}\,\xi^{\mu} - {}_o\underset{\sim}{T}_A\,\xi^{\mu}_{,\mu}\,. \tag{5.20}$$

In (5.19) and (5.20) we have replaced the collection of indices $\left({}^{\alpha_1\alpha_2\cdots}_{\beta_1\beta_2\cdots}\right)$ by capital letters as indicated. The ω's appearing in the last two equations are constants which may be calculated for any collection of indices.

22. Second, Third and Fourth Operator Identities

Let us consider a Lagrangian density function $\underset{\sim}{L}(f_A)$ of a set of operators f_A. If we subject the f_A to a variation δf_A, then

$$\delta \underset{\sim}{L}(f_A) = F^A_B\, \delta f_A\, G^B, \tag{5.21}$$

where F^A_B and G^B are operators in general. These latter quantities can be calculated if the form of $\underset{\sim}{L}(f_A)$ is given. Here again the capital letters refers to collection of indices and the summation convention is used.

Now, if $\underset{\sim}{L}$ can be expressed as a sum of polynomial functions of the operators f_A we have upon noting that for any operator Γ

$$[\Gamma, f_{A_1} f_{A_2} \cdots f_{A_N}] = [\Gamma, f_{A_1}] f_{A_2} \cdots f_{A_N} + f_{A_1} [\Gamma, f_{A_2}] f_{A_3} \cdots f_{A_N} + \cdots,$$

while

$$\delta f_{A_1} f_{A_2} \cdots f_{A_N} = (\delta f_{A_1}) f_{A_2} \cdots f_{A_N} + f_{A_1} (\delta f_{A_2}) f_{A_3} \cdots f_{A_N} + \cdots$$

that*

*See (5.39) for more general argument and a simpler and more useful version.

$$[\Gamma, \underset{\sim}{L}] \equiv F^A_B [\Gamma, f_A] G^B \tag{5.21a}$$

(5.21a) will be referred to as the second identity.

If we introduce $\underset{\sim}{L}$ in place of L in (5.1) we conclude by comparison with (5.2) and (5.5) upon using the cyclic property of the trace that the first identity takes on the form

$$[f_A, G^B F^A_B] \equiv 0, \tag{5.22}$$

in terms of the F^A_B and G^B appearing in (5.21). (5.22) is an identity and does not depend upon any particular physical interpretation.

Now let us consider the invariant

$$_\circ I \equiv \int_V \underset{\sim}{L}(_\circ C)\, d^4{}_\circ C, \tag{5.23}$$

where we have denoted by $_\circ\underset{\sim}{L}(_\circ C)$ the expression

$$\begin{aligned} \underset{\sim}{L}(_\circ C) &\equiv e^{i_\circ C^\mu{}_\circ p_\mu} \underset{\sim}{L}(f_A) e^{-i_\circ C^\mu{}_\circ p_\mu} \\ &= \underset{\sim}{L}(_\circ f_A(_\circ C)), \end{aligned} \tag{5.24}$$

where quantities with symbol $_\circ(\ )$ denote the Lorentz (flat space)

components. (5.23) in conjunction with (5.24) is an example of integration of operators as exemplified by (5.6). Now if we take the first variation of (5.23) we have upon using (5.21), where $\underset{\sim}{L}(c)$ is obtained from ${}_o\underset{\sim}{L}({}_oc)$ via (5.10),

$$\begin{aligned} \delta I &= \int_v \delta \underset{\sim}{L}(c)\, d^4 c, \\ &= \int_v F^A_B(c)\, \delta f_A(c)\, G^B(c)\, d^4 c. \end{aligned} \tag{5.25}$$

If our variations are due to an arbitrary variation of our c- number mesh system, then according to (5.18)

$$\delta_m I = -\int_v \left({}_o\underset{\sim}{L}({}_oc)\, \xi^\mu\right)_{,\mu} d^4\, {}_oc, \tag{5.26}$$

which may be transformed into a surface integral

$$\delta_m I = -\int_v {}_o\underset{\sim}{L}({}_oc)\, \xi^\mu\, d\sigma_\mu({}_oc). \tag{5.27}$$

Consequently, if the arbitrary c- numbers ξ^μ appearing in (5.12) are chosen such as to vanish on the surface σ then $\delta_m I = 0$. This corresponds to the circumstance that our c- number mesh system is chosen to be subject to infinitesimal deformations within the surface

σ bounding V and vanishing on the surface.

But the second equation of (5.25) in conjunction with (5.19) with $T_A \rightarrow f_A$, implies that

$$\delta_m I =$$

$$-\int_V \Big\{ W^{C\mu}_{A\nu}\, {}_0F^A_B({}_0C)\, {}_0f_C({}_0C)\, {}_0G^B({}_0C)\, \xi^\nu_{,\mu}$$

$$+ {}_0F^A_B({}_0C)\, {}_0f_A({}_0C)\, {}_{0,\mu} G^B({}_0C)\, \xi^\mu \Big\}\, d^4\, {}_0C. \qquad (5.28)$$

The above may be written as

$$\delta_m I =$$

$$\int_V \Big\{ W^{C\mu}_{A\nu} \big[{}_0F^A_B({}_0C)\, {}_0f_C({}_0C)\, {}_0G^B({}_0C) \big]_{,\mu} -$$

$${}_0F^A_B({}_0C)\, {}_0f_A({}_0C)_{,\nu}\, {}_0G^B({}_0C) \Big\}\, \xi^\nu\, d^4\, {}_0C$$

$$-\int_\sigma \big[W^{C\mu}_{A\nu}\, {}_0F^A_B({}_0C)\, {}_0f_C({}_0C)\, {}_0G^B({}_0C) \big]\, \xi^\nu\, d\sigma_\mu({}_0C). \qquad (5.29)$$

Now if our arbitrary infinitesimal deformation is chosen to vanish on the boundary σ bounding V then (5.27) shows that $\delta_m I = 0$,

which would imply in view of (5.29) that only the volume integral would survive which in turn would have to vanish on account of the arbitrariness of the deformation vector ξ^{ν} within the volume. We conclude, consequently, that

$$\omega^{C\mu}_{A\nu}\left[{}_{0}F^{A}_{B}({}_{0}C)\,{}_{0}f_{C}({}_{0}C)\,{}_{0}G^{B}({}_{0}C)\right]_{,\mu}$$

$$-\,{}_{0}F^{A}_{B}({}_{0}C)\,{}_{0}f_{A}({}_{0}C)_{,\nu}\,{}_{0}G^{B}({}_{0}C) \equiv 0. \qquad (5.30)$$

(5.30) is an identity. But all of the symbols in (5.30) which involve $({}_{0}C)$ are obtained via unitary transformations of the type (5.24). Consequently, (5.30) may be written as (the third identity).

$$i\,\omega^{C\mu}_{A\nu}\left[p_{\mu}, F^{A}_{B}\,f_{C}\,G^{B}\right] - i\,F^{A}_{B}\left[p_{\nu}, f_{A}\right]G^{B} \equiv 0. \qquad (5.31)$$

Again we emphasize that (5.31) is an identity. (5.31) and (5.22) are the principal identities of this piece of work and have as yet nothing whatsoever to do with physics. We shall presently make contact with physics after considering an example.

Let us consider the operator

$$\underset{\sim}{L} \equiv A^{\mu} B_{\mu} \underset{\sim}{C},$$

where A^{μ} and B_{μ} are respectively contravariant and covariant

vectors while $\underset{\sim}{C}$ is a scalar density.

$$\delta \underset{\sim}{L} = (\delta A^{\mu}) B_{\mu} \underset{\sim}{C} + A^{\mu}(\delta B_{\mu}) \underset{\sim}{C} + A^{\mu} B_{\mu} (\delta \underset{\sim}{C}).$$

If the variations are of the type given by (5.15) and (5.18) then the above may be written as

$$\begin{aligned}
\delta \underset{\sim}{L} = & +(A^{\sigma} \xi^{\mu}{}_{,\sigma} - A^{\mu}{}_{,\sigma} \xi^{\sigma}) B_{\mu} \underset{\sim}{C} \\
& - A^{\mu}(B_{\sigma} \xi^{\sigma}{}_{,\mu} + B_{\mu,\sigma} \xi^{\sigma}) \underset{\sim}{C} \\
& - A^{\mu} B_{\mu} (\underset{\sim}{C} \xi^{\sigma})_{,\sigma} \\
= & -(A^{\mu} B_{\mu})_{,\sigma} \underset{\sim}{C} \xi^{\sigma} - (A^{\mu} B_{\mu})(\underset{\sim}{C} \xi^{\sigma})_{,\sigma} \\
= & -(A^{\mu} B_{\mu} \underset{\sim}{C} \xi^{\sigma})_{,\sigma} \quad ,
\end{aligned}$$

so that in the volume integral of $\delta \underset{\sim}{L}$ (compare with 5.29) has for the coefficient of ξ^{ν} zero, which we wished to show. To illustrate (5.22) we have on comparison with (5.2) and (5.5) to show that

$$[A^{\mu}, B_{\mu} \underset{\sim}{C}] + [B_{\mu}, \underset{\sim}{C} A^{\mu}] + [\underset{\sim}{C}, A^{\mu} B_{\mu}] = 0.$$

Evaluation of the commutators show that such is indeed the case. These examples should give us confidence regarding the general results obtained, namely those summarized in (5.31) and (5.22).

As a summary we have from (5.22), (5.21a) and (5.31) the first, second, and third identities given by

$$[f_A, G^B F^A_B] \equiv 0,$$

$$[\Gamma, \underset{\sim}{L}] \equiv F^A_B [\Gamma, f_A] G^B, \qquad (5.32)$$

$$i\,\omega^{C\mu}_{A\nu}[p_\mu F^A_B f_C G^B] - i F^A_B [p_\nu, f_A] G^B \equiv 0,$$

respectively, where Γ is any operator and F^A_B and G^B are defined through (5.21)

$$\delta \underset{\sim}{L} \equiv F^A_B \,\delta f_A\, G^B, \qquad (5.33)$$

and further where the ω's are defined by (5.19)

$$\delta_m f_A = -\omega^{B\mu}_{A\nu}\, {}^{\circ}f_B\, \xi^\nu_{,\mu} - {}^{\circ}f_{A,\mu}\, \xi^\mu . \qquad (5.34)$$

(5.34) shows that $\omega^{C\mu}_{A\nu} F^A_B f_C G^B$ appearing in the third

equation of (5.41) is merely the coefficient of $-\xi^{\mu}_{,\nu}$ in $\delta \underset{\sim}{L}$ given by (5.33) if the variations arise simply from an infinitesimal C-number transformation given by (5.12), namely

$$\delta_C f_A \equiv - W^{B\mu}_{A\nu} f_B \, \xi^{\nu}_{,\mu} . \tag{5.35}$$

Upon combining the second and third identities of (5.32) with Γ replaced by p_ν, we conclude that

$$i\left[p_\mu , F^A_B W^{C\mu}_{A\nu} f_C G^B - \delta^{\mu}_{\nu} \underset{\sim}{L}\right] \equiv 0. \tag{5.36}$$

These identities especially those obtained using the transformation properties of tensors are useful. However, we had to appeal to our integration formalism and there may be simpler ways to obtain the results independently of any reference to our integration formalism. The first expression of (5.32) is quite general, the second expression can be put in a more useful form by noting that

$$\delta \, \mathrm{Tr}\, \Gamma \underset{\sim}{L} = \mathrm{Tr}\,(\Gamma \delta \underset{\sim}{L} + \underset{\sim}{L}\, \delta \Gamma), \tag{5.37}$$

which from (5.21) and cyclic property of the trace may be written as

$$\delta \, \mathrm{Tr}\, \Gamma \underset{\sim}{L} = \mathrm{Tr}\,(\underset{\sim}{L}\, \delta \Gamma + G^B \Gamma F^A_B \, \delta f_A), \tag{5.38}$$

so that we conclude upon comparison with the work which led to (5.22) that (the fourth identity)

$$[\Gamma, \underset{\sim}{L}] \equiv -[f_A, G^B \Gamma F^A_B], \qquad (5.39)$$

for any operator Γ . This result is independent of the customary consideration regarding the invariance properties of $\underset{\sim}{L}$ and could be made the starting point for construction of Charge-Current Densities and Stress-Energy-Momentum-Tensors. In the following section we shall use (5.39) as starting point and determine what conditions must be satisfied for the existence of Conservation equations.

23. Question of the Existence of Conservation Equations

In certain cases it is not too difficult to construct non-local stress-energy momentum tensors by inspection or by comparison with well known local field expressions for the stress-energy-momentum tensors. For these cases the operator field equations involve commutators of the fields with the displacement operators f_μ . The question arises as to the construction of the stress-energy-momentum tensor when the operator field equations involve the operator fields and the displacement operator in an arbitrary way. Some time ago[3] an identity was discovered which had as a result shown the existence of a non-local vector function N^μ which satisfied a conservation

equation $[p_\mu, N^\mu]$. We may be able to use a generalization of this identity (the fourth identity) (5.48) to study the structure of conservation equations.

From (5.22) and (5.39) we have upon replacing Γ by p^ν in the latter the two equations

$$\begin{aligned} &[f_A, G^B F^A_B]_- \equiv 0, \\ &[p^\nu, L]_- + [f_A, G^B \Gamma F^A_B]_- \equiv 0. \end{aligned} \tag{5.40}$$

Now if we identify $f_\mu, \mu = 0, 1, 2, 3,$ to be the displacement operators p_μ and the rest of the A's to be collection of indices specifying the fields, say A', then (5.40) may be expressed as

$$\begin{aligned} &[p_\mu, G^B F^\mu_B]_- \equiv -[f_{A'}, G^B F^{A'}_B]_-, \\ &[p, \eta^{\mu\nu} L + G^B p^\mu F^\nu_B]_- \equiv -[f_{A'}, G^B p^\nu F^{A'}_B], \end{aligned} \tag{5.41}$$

where $\eta^{\mu\nu}$ is the flat space metric tensor. The first equation of (5.41) would imply the existence of the conservation equation

$$[p_\mu, N^\mu]_- = 0, \tag{5.42}$$

if the field equations

$$G^B F^{A'}_B = 0, \tag{5.43}$$

are satisfied, where

$$N^\mu \equiv G^B F^\mu_B . \tag{5.44}$$

On the other hand if (5.43) is satisfied, we cannot in general be assured that

$$[p_\mu, T^{\mu\nu}]_- = 0, \tag{5.45}$$

unless

$$[f_{A'}, G^B p^\nu F^{A'}_B]_- = 0, \tag{5.46}$$

where the stress energy-momentum tensor is

$$T^{\mu\nu} \equiv \eta^{\mu\nu} L + G^B p^\mu F^\nu_B . \tag{5.47}$$

(5.46) and (5.43) may be simultaneously satisfied if the fields are local

fields. Another version of (5.46) can be written as

$$[f_{A'}, G^B p^\nu F_B^{A'}]_- = [f_{A'}, G^B [p^\nu, F_B^{A'}]]_-, \quad \text{or}$$
$$= [f_{A'}, [G^B, p^\nu]_- F_B^{A'}]_-, \tag{5.48}$$

if (5.43) is satisfied.

We conclude that there exists a vector operator N^μ (5.44) and a tensor operator $T^{\mu\nu}$ (5.47) which satisfy conservation equations (5.42) and (5.45) if

$$G^B F_B^{A'} = 0, \quad \text{(Field Equations)} \tag{5.49}$$

implies

$$[f_{A'}, G^B p^\nu F_B^{A'}]_- = 0, \tag{5.50}$$

also. It does not seem to be possible to make any further significant reductions on this phase of the problem.

Application of this method to the simple case of the Lagrangian $L_\pm$ given by

$$4L_\pm = [p^\mu, U]_\pm [p_\mu, U]_\pm, \tag{5.51}$$

which involves a single scalar field U and commutator or anti-commutator expressions

$$[A, B]_{\pm} \equiv AB \pm BA, \tag{5.52}$$

shows that if (5.49) is satisfied, namely,

$$[p^{\mu}, [p_{\mu}, U]_{\pm}] = 0,$$

then (5.50) is satisfied. $T_{\pm}^{\mu\nu}$ turns out to be

$$4\, T_{\pm}^{\mu\nu} = \eta^{\mu\nu} [p_{\alpha}, U]_{\pm} [p^{\alpha}, U]_{\pm} - [p^{\mu}, U]_{\pm} [p^{\nu}, U]_{\pm} - [p^{\nu}, U]_{\pm} [p^{\mu}, U]_{\pm}, \tag{5.53}$$

while N^{μ} works out to be

$$N_{\pm}^{\mu} = [U, [U, p^{\mu}]_{\pm}]_{\pm} \tag{5.54}$$

VI. PHYSICAL INTERPRETATION OF OPERATOR FIELDS

24. Introduction

It is appropriate to close this initial phase of the investigation of the theory of operator fields by suggesting a possible interpretation of the $C-$ numbers appearing in the expansion of operator fields of type utilizing the basic functions $\chi_{n'k'}$ defined by (2.37): in particular the $C-$ number n' which in the following we shall denote by q. A clue regarding an interpretation is given to us by examining the result of a novel feature of a variation principle for non-local fields (5.42, 5.44)[3] whereby the existence of a divergence-less four vector is shown. It was also shown that it is possible for a (real) hermitian non-local field to give rise to an operator for the net number of quanta with either positive or negative integer eigenvalues.[4] This property suggests that even a real field in operator field theory has associated with it a four-current vector. Only for complex fields is it possible to construct four current vectors with the customary properties for local fields in present theories. Since even real non-local fields imply the existence of particles with charge, one would expect that such a field could be in interaction with external local electric and magnetic fields. The structure of the interaction must be chosen in such a way as to be non-existent in the limit of the local field case. This would imply that even a real non-local field has the

potentiality of manifesting properties of non-real local fields in interaction with an external E, H field. For the sake of simplicity the case for a constant external electromagnetic field is considered, while the non-local field is taken to be of the electromagnetic type.

One way to introduce an interaction of the type contemplated above is to replace the displacement operator p_μ by the operator

$$P_\mu = p_\mu - g/c\, A^e_\mu, \tag{6.1}$$

where g is a coupling constant, c the velocity of light and A^e_μ the external local vector potential. Such a replacement assures us that the commutator expression

$$[P_\mu, U]_- = [p_\mu, U]_- - g/c\, [A^e_\mu, U]_-, \tag{6.2}$$

is devoid of terms containing the coupling constant g if U is a local function. As a consequence of the definition of P_μ we find that the P_μ do not commute amongst themselves:

$$\begin{aligned}[P_\mu, P_\nu]_- &= -g/c\,[p_\mu, A^e_\mu] - g/c\,[A^e_\mu, p_\nu] \\ &= -\hbar\, g/ic\, F^e_{\mu\nu}, \end{aligned} \tag{6.3}$$

where $F^{e}_{\mu\nu}$ is the assymmetric external electromagnetic field strength tensor.

25. The Field Equations

The field equations for our non-local electromagnetic field in interaction with a given external local E, H field may be obtained in the same manner as the commutator equations for an electromagnetic field in an earlier piece of work[5] by merely replacing the p_μ by P_μ in the commutator equations for Maxwell's Equations to obtain

$$\frac{i}{\hbar}[P_\mu, F^{\mu\nu}] = 0, \tag{6.4}$$

$$\frac{i}{\hbar}[P_\mu, F_{\nu\sigma}] + \frac{i}{\hbar}[P_\nu, F_{\sigma\mu}] + \frac{i}{\hbar}[P_\sigma, F_{\mu\nu}] = 0, \tag{6.5*}$$

where

$$F_{\mu\nu} \equiv \frac{i}{\hbar}[P_\mu, A_\nu] - \frac{i}{\hbar}[P_\nu, A_\mu]. \tag{6.6}$$

The above equations reduce to the ordinary equations if the A's are local independently of the value assigned to the coupling constant.

Now let us consider the case where the A^{e}_{μ} are the vector potentials for a constant local external E, H field. For such a case

*(6.5) is only valid for a constant E, H field. In general this equation would be non-homogeneous.

the A^{ℓ}'s may be written as

$$A^{\ell}_{\mu} = C_{\mu\alpha} \chi^{\alpha}, \tag{6.7}$$

where the $C_{\mu\alpha}$'s are $C-$ numbers depending upon the external field strengths E and H, and the χ's are the space time operators. The proper identification of the $C_{\mu\nu}$'s which lead to the constant E, H field follow upon making the identification

$$\begin{aligned} C_{\mu 0} &= 0, \quad \mu = 0, 1, 2, 3 \\ C_{0s} &= F^{\ell}_{0s}, \quad s = 1, 2, 3 \\ C_{rs} &= -C_{sr} = \tfrac{1}{2} F^{\ell}_{rs}, \quad r, s = 1, 2, 3 \end{aligned} \tag{6.8}$$

where the F^{ℓ}'s are the constant external field strengths. For the case under consideration here we conclude that $[P_{\mu}, P_{\nu}]$ is a $C-$ number as we see from (6.3). This observation coupled with (6.4) enables us to write

$$[P_{\mu}, [P^{\mu}, A^{\nu}]] = 0, \tag{6.9}$$

providing we take as our supplementary condition on the A's

$$[P_\mu, A^\mu] = 0. \tag{6.10}$$

Now let us seek a solution of (6.9) of the form of a non-local plane wave defined by

$$A^\mu_{q,k} \equiv a^\mu e^{iq^\nu p_\nu/\hbar} e^{ik_\nu x^\nu/\hbar}, \tag{6.11}$$

where a^μ, q^ν and k_ν are C- numbers. The introduction of (6.11) in (6.9), (6.10) and (6.5) lead to the equations restricting the above C- numbers if we make the observation that

$$\frac{i}{\hbar}[P_\mu, e^{iq^\nu p_\nu/\hbar} e^{ik_\nu x^\nu/\hbar}] =$$

$$(k_\mu + \tfrac{e}{c}\, \varsigma_{\mu\alpha}\, q^\alpha) e^{iq^\nu p_\nu/\hbar} e^{ik_\nu x^\nu/\hbar}, \tag{6.12}$$

which is a consequence of our fundamental commutation relationship

$$[x^\mu, p_\nu] = i\hbar\, \delta^\mu_\nu. \tag{6.13}$$

The result (6.12) implies from (6.10) and (6.9) the following relation-

ships existing between the C- numbers a^μ, q^ν and k_ν :

$$K_\mu = (k_\mu + g/c\; g_{\mu\alpha}\, q^\alpha), \tag{6.14}$$

$$a^\mu K_\mu = 0, \tag{6.15}$$

$$K^\mu K_\mu = 0. \tag{6.16}$$

From (6.6) we may write

$$F_{\mu\nu} = f_{\mu\nu}\, e^{i q^\nu p_\nu/\hbar}\, e^{i k_\nu x^\nu/\hbar}, \tag{6.17}$$

where the C- numbers $f_{\mu\nu}$ are given by

$$f_{\mu\nu} = K_\mu a_\nu - K_\nu a_\mu . \tag{6.18}$$

These results enable us to state as for the classical treatment of Maxwell's equation that upon identifying $F_{\mu\nu}$ according to the scheme

$$\begin{gathered} F_{s0} = -F_{0s} = -c\, E_s, \qquad s = 1, 2, 3 \\ -F_{21} = F_{12} = -H_3,\; -F_{31} = F_{13} = H_2,\; F_{23} = -F_{32} = -H_1, \end{gathered} \tag{6.19}$$

the vector K_s , $s=1,2,3$ is perpendicular to the vector E_s and H_s . Furthermore, upon using the scheme (6.19) with the F's replaced by f's and E_s by e_s and H_s by h_s we find that the vector e_s is perpendicular to h_s all of which is in complete agreement with the properties of a plane wave solution of Maxwell's equations with the exception of the finding that the ordinary propagation four-vector k_μ is to be replaced by K_μ given by (6.14). The general solution of (6.9) will of course correspond to a sum over q and k subject to the restrictions (6.14), (6.15) and (6.16). The coefficient of a^μ in (6.11) forms a set of basic functions aside from a normalization factor.[6] These are essentially the $X_{n'k'}$ given by (2.37).

26. Mass Equivalent of Non-Local Photon

The results that we have obtained would enable us to consider the polarization effects of the non-local electromagnetic field in interaction with the constant external E^a, H^a field. It would be more interesting, however, to consider the consequences of the relationship (6.16). If we define the equivalent mass μ through the equation

$$\mu^2 c^4 = k_0^2 - c^2 |k|^2, \tag{6.20}$$

then (6.20) in conjunction with (6.14) and (6.16) implies that

$$\mu^2 c^4 = b_0^2 + c^2 \underset{\sim}{b} \cdot \underset{\sim}{b} + 2c^2 \underset{\sim}{b} \cdot \underset{\sim}{k} \pm 2c\, b_0 |\underset{\sim}{b} + \underset{\sim}{k}| , \tag{6.21}$$

where

$$\left.\begin{aligned} b_0 &\equiv g/c \; C_{0\alpha}\, q^\alpha = -g \underset{\sim}{E}^{e} \cdot \underset{\sim}{q} \\ \underset{\sim}{b} &\equiv g/2c \,(\underset{\sim}{H}^{e} \times \underset{\sim}{q}) = -g/2c\,(\underset{\sim}{q} \times \underset{\sim}{H}^{e}) \end{aligned}\right\} \tag{6.22}$$

in vector notation.

(6.21) together with (6.20) or direct appeal to (6.14) yields

$$k_0 = -b_0 \pm c\,|\underset{\sim}{k} + \underset{\sim}{b}| , \tag{6.23}$$

which may be interpreted to indicate that when a non-local photon is in an electric field there is a contribution to the energy of the nature of a dipole term, namely $-b_0 = g \underset{\sim}{E}^{e} \cdot \underset{\sim}{q}$, so that we may attribute to a non-local photon the electric dipole moment $g \underset{\sim}{q}$. This identification has as a consequence given physical significance to the C- numbers q appearing in our expression for a non-local plane wave. It would appear then that a non-local photon would be

subject to deflections in an inhomogeneous electric field as a consequence of its electric dipole moment and the massive properties which it would possess by virtue of (6.21) in the presence of a constant electric field and consequently also in an inhomogeneous one. To consider the effect of our H^e field, we may expand k_0 given by (6.23) in a power series in the H^e's. Upon retaining terms which contain H^e to the first power and making use of (6.22) we find

$$k_0 \cong g\,\underset{\sim}{E}^e \cdot \underset{\sim}{q} \pm \frac{g}{2}\,\underset{\sim}{H}^e \cdot (\underset{\sim}{q} \times \underset{\sim}{k})/|k| \pm c\,|k|. \qquad (6.24)$$

If we take $k_0 > 0$ in the absence of the E, H field, (6.24) indicates that our non-local photon possesses a magnetic moment: $(g/2)(\underset{\sim}{q} \times \underset{\sim}{k})/|k|$. Thus, we would also expect a non-local photon to be subject to a deflection in an inhomogeneous magnetic field. We note that the dipole moment and magnetic moment are perpendicular to one another while the propagation direction $\underset{\sim}{k}/|k|$ is perpendicular to the magnetic moment vector.

In general we will not be able to say that $\mu^2 > 0$ unless certain conditions are satisfied. We shall investigate the situation on the basis of the expression (6.24) for k_0 which provides us with first order correction terms in terms of the first powers of our external E, H field. Upon introducing (6.24) into (6.20) we obtain

to the desired order (the upper signs in (6.24) is taken),

$$\mu^2 c^4 \simeq c|k|g\left[2\,\mathbf{E}^e\cdot\mathbf{q} + \mathbf{H}^e\cdot(\mathbf{q}\times\mathbf{k})/|k|\right],$$
$$\simeq c|k|g\,\mathbf{q}\cdot\left[2\,\mathbf{E} + \mathbf{k}\times\mathbf{H}^e/|k|\right]. \qquad (6.26)$$

Consequently, in order that $\mu^2 > 0$, the dipole moment vector $g\,\mathbf{q}$ must be orientated in such a way as to be intermediate to those positions corresponding to the extreme cases of perpendicularity and isodirectionality to the vector $\mathbf{F}$ defined by

$$\mathbf{F} \equiv c|k|\left[2\,\mathbf{E}^e + \mathbf{k}\times\mathbf{H}^e/|k|\right]. \qquad (6.27)$$

In the former case μc^2 would attain its smallest value zero while in the latter it would have its maximum value, namely $\sqrt{|g\mathbf{q}||F|}$. If we average μ over those directions for which $\mu^2 > 0$ we obtain

$$\langle\mu\rangle = \frac{2}{3c^2}\left(|g\mathbf{q}||F|\right)^{1/2}$$
$$= \tfrac{2}{3}c^2\left(c|g||\mathbf{q}||\mathbf{k}|\left|2\,\mathbf{E}^e + \mathbf{k}\times\mathbf{H}^e/|k|\right|\right)^{1/2}, \qquad (6.28)$$

which would provide us with an estimate of the equivalent mass of a

non-local photon in an external E, H field.

Let us now make an estimate as to the order of magnitude of $\langle \mu \rangle$. Now, upon making the substitution

$$|\underset{\sim}{F}|/c|k| = |2\underset{\sim}{E}^e + \underset{\sim}{k} \times H^e/|k||, \tag{6.29}$$

in (6.28) to indicate the magnitude of the external effects of the field as measured in Gaussian units we find that

$$\langle \mu \rangle / m = 4/3\,(\pi\beta/mc^2)^{1/2}\,|g/e|^{1/2}\,|q/\lambda|^{1/2}\,(|F|/|k|c)^{1/2}, \tag{6.30}$$

where e is the electron charge, m its mass, β the magnetic moment of a Bohr magneton, and λ the wave length of our non-local photon. However, we must have on the basis of the assumption of the validity of the expansion which lead to (6.24)

$$2\pi\,|g/e|\,|q/\lambda|\,(\lambda/\lambda_c)^2\,\beta/mc^2\,(|F|/|k|c) < 1, \tag{6.31}$$

where λ_c denotes the Compton wave length and we have assumed the rough numerical equivalence between $|\underset{\sim}{F}|$ and $c\,|\underset{\sim}{k} \times \underset{\sim}{H}^e|$. If we take $\lambda \simeq 10^{-13}$, $g \simeq e$ and $|\underset{\sim}{F}|/|k|c \simeq 10^4$ Gauss, (6.31) implies that if $q < 10{,}000$ cms. the expansion (6.24) is valid.

These results would show that if we take $\mathfrak{g}$ to be the order of magnitude of a Compton wave length that $\langle\mu\rangle \simeq 10^{-3}$ electronic masses which would correspond ~ 500 E. V. On the other hand if $\mathfrak{g}$ is of the order of an electronic radius $\langle\mu\rangle \simeq 3 \times 10^{-5}$ electronic masses $\Leftrightarrow \sim 15$ E. V.

27. Equivalent "Index of Refraction" of E, H Region

In order to gain a rough idea as to what could be expected to occur when anon-local photon impinges upon a region possessing an E, H field we may assign to the region an equivalent index of refraction

$$\eta \equiv c|k|/k_0$$

$$\simeq 1 - g\,\underset{\sim}{\mathfrak{g}} \cdot \underset{\sim}{F}/2c^2|k|^2 . \tag{6.32}$$

(6.32) is obtained by making an analogy with optical theory and using our expressions for k_0 given by (6.24) which in turn may be expressed in terms of $\underset{\sim}{F}$ defined in (6.27). (6.32) indicates that if $\mu^2 > 0$, the "wave velocity" would be greater in a region containing the E, H fields than in one devoid of such fields. Moreover, we note a dependence of η upon the direction of propagation

relative to the orientation of our external E, H field and upon the wave length which enters into our expression (6.32) in such a way as to show that one can expect η to deviate from unity in a manner directly proportional to the wave length. If $\mu^2 < 0$, then $\eta > 1$ which implies that the "wave velocity" is greater in a region not containing the E, H field. In this case also, the region would possess anisotropic properties with η deviating from unity linearly with increasing wave length.

28. Summary

In order to indicate conveniently the order of magnitude of the effects implied by the existence of a non-local photon in interaction with a constant external E, H field, let us define

$$(\underset{\sim}{q} \cdot \underset{\sim}{E}) / c\,|\underset{\sim}{k}|\,|\underset{\sim}{q}| \equiv B, \tag{6.33}$$

as measured in Gaussian units and put

$$\frac{g\,|\underset{\sim}{q}|}{4\pi\beta} \equiv \gamma, \tag{6.34}$$

which is the number of Bohr magnetons $(\beta \equiv he/4\pi mc)$ expressed in terms of our coupling constant and the vector $\underset{\sim}{q}$ which appears in our non-local plane wave (6.11). The inequality (6.31) takes on the

form

$$\lambda \gamma B_{max} < 2mc^2/e, \tag{6.35}$$

where $B_{max.}$ denotes the value attained by B in (6.33) when $\underset{\sim}{q}$ is perpendicular to $\underset{\sim}{F}$ defined in (6.27). λ is the wave length of our non-local photon, m the electronic mass and e the charge. Introducing (6.33) and (6.34) into (6.30) and (6.32) we obtain

$$\langle\mu\rangle/m = 8\pi\beta/3c\,(me)^{-1/2}(\gamma B/\lambda)^{1/2}, \tag{6.36}$$

$$|n-1| = (e/2mc^2)\,\lambda\gamma B. \tag{6.37}$$

(6.35), (6.36), and (6.37) become upon introducing the numerical values of e, m, c, and β,

$$1.7\times 10^{3} > \lambda\gamma B_{max},$$

$$\langle\mu\rangle/m = 4.0\times 10^{-12}(\gamma B/\lambda)^{1/2}, \tag{6.38}$$

$$|n-1| = 3.2\times 10^{-4}(\lambda\gamma B).$$

(6.38) shows that for pronounced measurable optical effects, say

$|n-1| \cong 0.1$, some of the non-local photons would have to possess an equivalent magnetic moment of the order $\gamma = 300/\lambda B$ Bohr magnetons. If such were the case then $\langle\mu\rangle/m$ would attain a value of order $10^{-10}/\lambda$ electron masses.

These effects, of course, would vanish in the limiting case of local fields: $g_s = 0$, independently of the value assigned to the coupling constant g. Consequently, it would appear that non-local field theories could be examined in the light of experiments which are implied in this work if we assume that our procedure for introducing an interaction is correct and that the photons existing in nature are non-local. In the same way, we can consider other non-local "particles": "neutral" or "charged" in interaction with a constant E, H field of the type considered here.

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